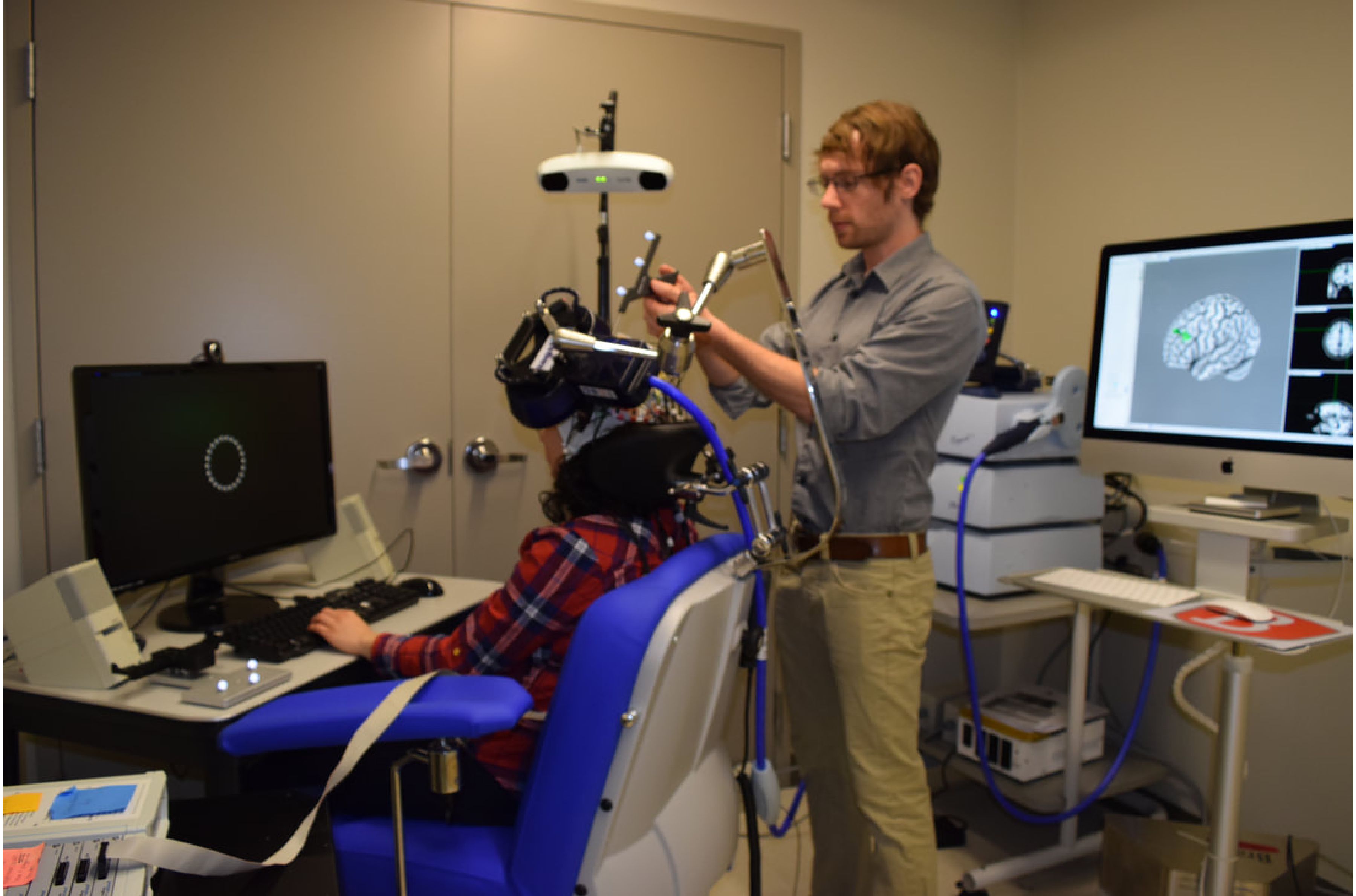
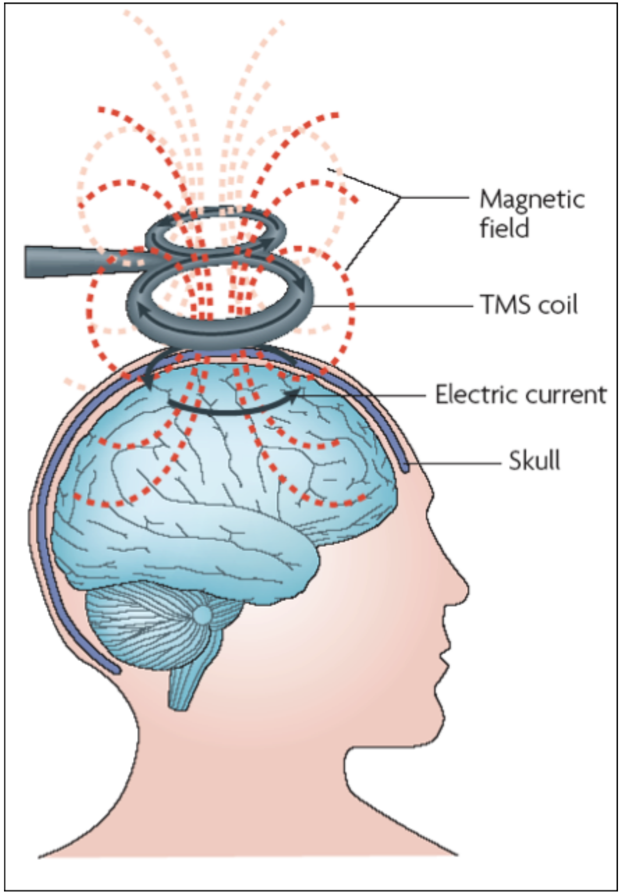
Cyclicity Measures and Causality Inference in Neural Time Series

Chad Giusti
Department of Mathematical Sciences
University of Delaware

Abel Symposium 2018



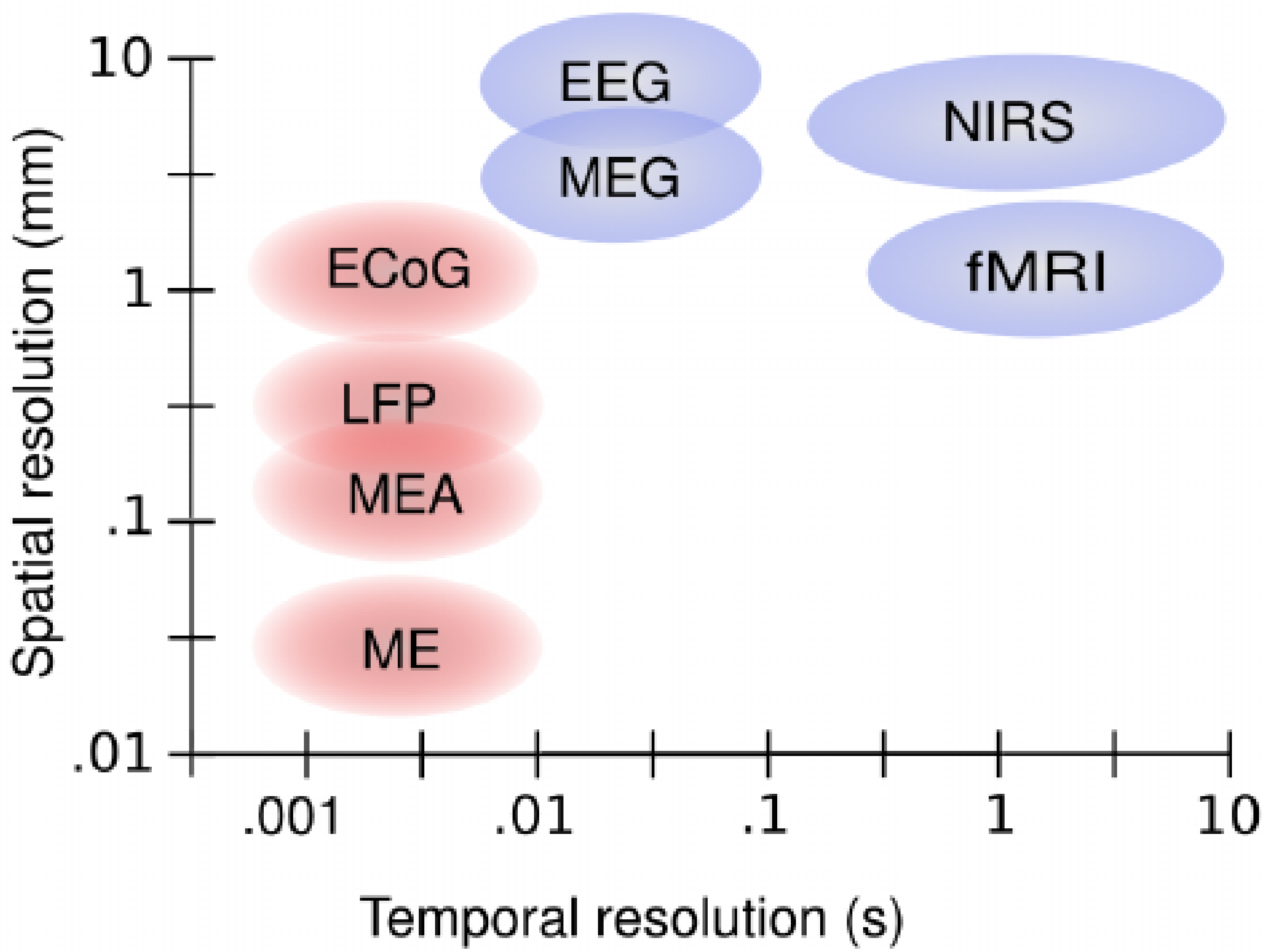
TMS: Non-invasive Neural Stimulation



Electromagnetic "wand" applied to the scalp.

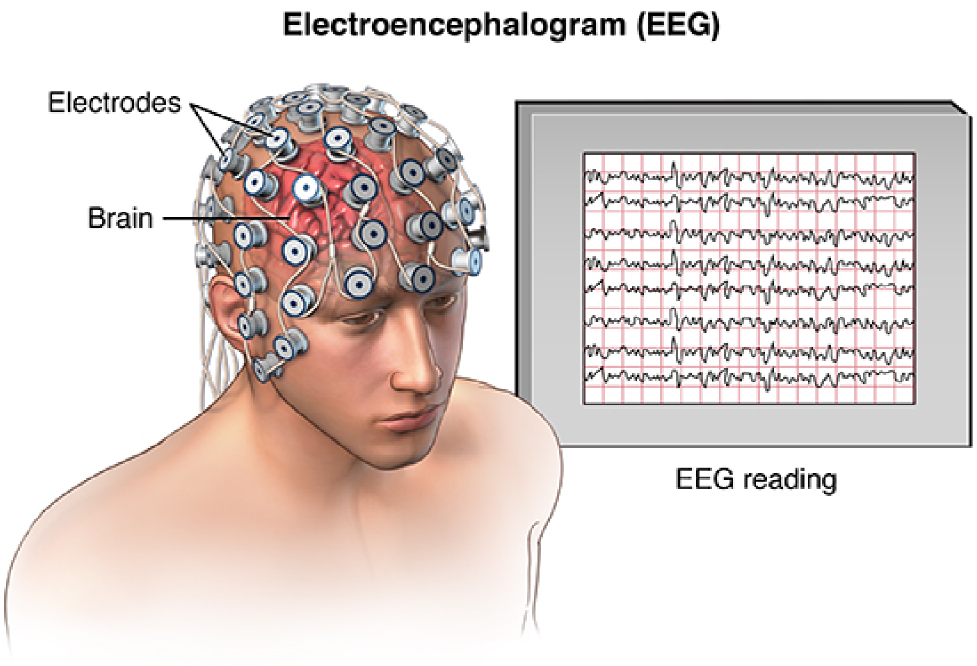
- Pulse occurs on the ~1 ms time scale
- Induces activity in ~ 1 cm^3 of cortex
- Activity returns to baseline in < 500 ms

Neurostimulation has been empirically shown to have significant impact on cognitive processes (memory, executive function, task learning) and clinical applications (traumatic brain injury, aphasia, physical therapy).



EEG
MEG
NIRS
fMRI
ECoG
LFP
MEA
ME
10
1
.1
.01
.001
.01
.1
1
10
Spatial resolution (mm)
Temporal resolution (s)

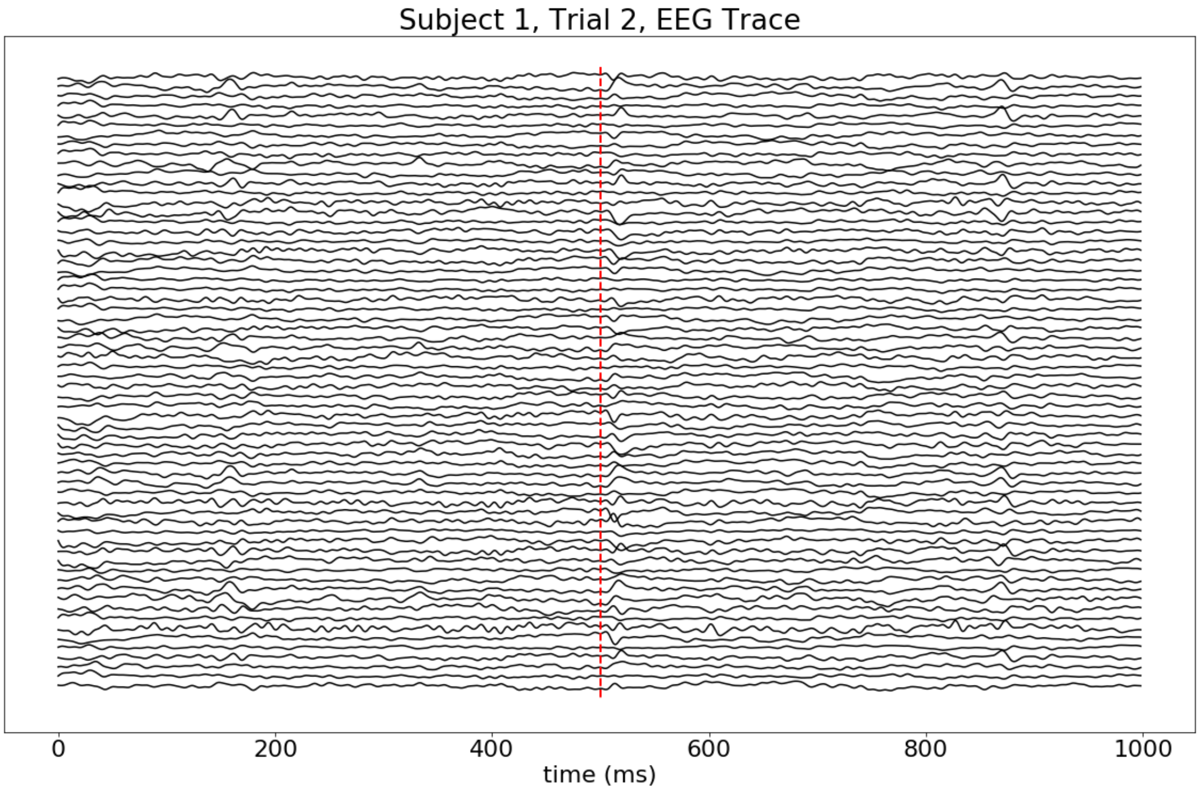
EEG: High-Speed Neural Activity Recording



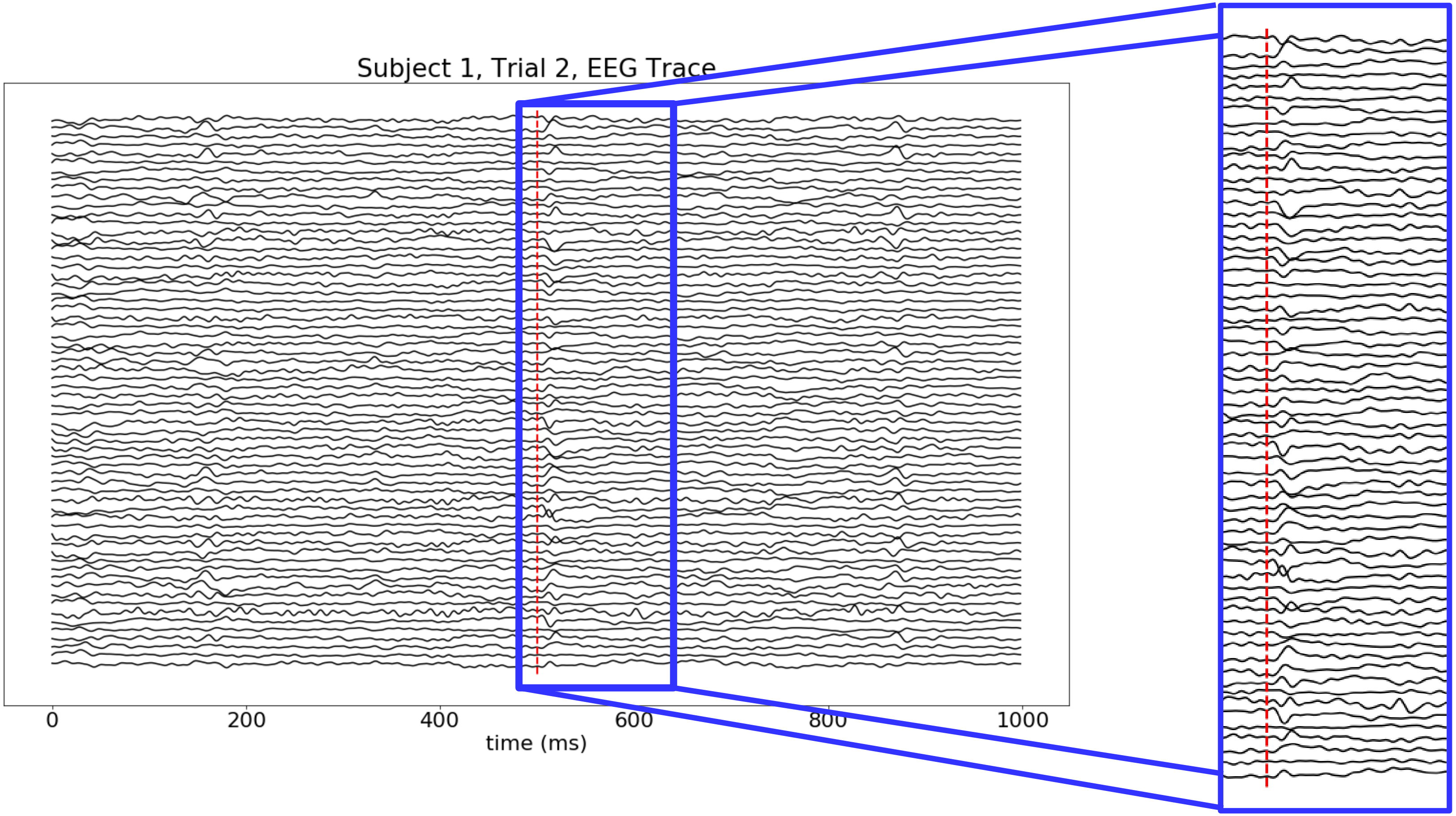
Electrodes applied to the scalp
- Usually 32 or 64 channels
- Can be integrated with TMS

EEG is inexpensive, minimally invasive and in widespread use in both clinical and experimental contexts.

What does the data look like?



What does the data look like?



The Questions

1) What's the difference between the EEG time series before and after stimulation in a single trial? Across many trials?

The Questions

1) What's the difference between the EEG time series before and after stimulation in a single trial? Across many trials?
2) If we change the details of the stimulation (location, duration, pattern) what effect does it have on the EEG time series?

The Questions

1) What's the difference between the EEG time series before and after stimulation in a single trial? Across many trials?
2) If we change the details of the stimulation (location, duration, pattern) what effect does it have on the EEG time series?
3) How do individuals in different populations (healthy vs. diseased, e.g.) respond to stimulation differently? How do we use the EEG data to predict these effects?

The Questions

1) What's the difference between the EEG time series before and after stimulation in a single trial? Across many trials?
2) If we change the details of the stimulation (location, duration, pattern) what effect does it have on the EEG time series?
3) How do individuals in different populations (healthy vs. diseased, e.g.) respond to stimulation differently? How do we use the EEG data to predict these effects?
4) Do the EEG time series before/after stimulation predict changes in cognitive measures or effects of stimulation in clinical application? How do we use this to plan treatment?

The Questions

1) What's the difference between the EEG time series before and after stimulation in a single trial? Across many trials?
2) If we change the details of the stimulation (location, duration, pattern) what effect does it have on the EEG time series?
3) How do individuals in different populations (healthy vs. diseased, e.g.) respond to stimulation differently? How do we use the EEG data to predict these effects?
4) Do the EEG time series before/after stimulation predict changes in cognitive measures or effects of stimulation in clinical application? How do we use this to plan treatment?

Current Methodology

Current Methodology

There aren't many options:

- Event Related Potentials
- Fourier Analysis
- Granger Causality
- Structural Equation Modeling (and various modifications)

Current Methodology

There aren't many options:

- Event Related Potentials
- Fourier Analysis
- Granger Causality
- Structural Equation Modeling (and various modifications)

Outside of ERP, these all suffer from assumptions of stationarity, linearity, and Gaussian noise, all of which are (very) false in this setting.

New Ideas Needed!

New Ideas Needed!

Think of this multichannel EEG data as a time series in $\mathbb{R}^n$.

$$f \in \mathrm{Map}([0,1], \mathbb{R}^n) = P\mathbb{R}^n$$

New Ideas Needed!

Think of this multichannel EEG data as a time series in $\mathbb{R}^n$.

$$f \in \mathrm{Map}([0,1], \mathbb{R}^n) = P\mathbb{R}^n$$

Numerical measurements of time series are just (zero-)cochains and we’re in luck, as algebraic topologists have already done a lot of hard work for us – and this work is currently being leveraged elsewhere!

Iterated Integrals: Cochains on Path Spaces

Let $\gamma : I \to \mathbb{R}^n$ and $\{\mathrm{d}x_i\}_{i=1}^n$ the standard basis for the 1-forms on $\mathbb{R}^n$.

Iterated Integrals: Cochains on Path Spaces

Let $\gamma : I \to \mathbb{R}^n$ and $\{\mathrm{d}x_i\}_{i=1}^n$ the standard basis for the 1-forms on $\mathbb{R}^n$.

For $i \in \{1, \ldots n\}$, define a new path, $S(\gamma)^i : I \to \mathbb{R}$ by

$$S(\gamma)(t)^i = \int_{s=0}^t \gamma^* \mathrm{d}x_i = \int_{s=0}^t \mathrm{d}\gamma_i = \gamma_i(t) - \gamma_i(0)$$

Iterated Integrals: Cochains on Path Spaces

Let $\gamma : I \to \mathbb{R}^n$ and $\{\mathrm{d}x_i\}_{i=1}^n$ the standard basis for the 1-forms on $\mathbb{R}^n$.

For $i \in \{1, \dots n\}$, define a new path, $S(\gamma)^i : I \to \mathbb{R}$ by

$$S(\gamma)(t)^i = \int_{s=0}^t \gamma^* \mathrm{d}x_i = \int_{s=0}^t \mathrm{d}\gamma_i = \gamma_i(t) - \gamma_i(0)$$

Iterate: Let $I = (i_1, i_2, \dots i_k)$, $i_\ell \in \{1, \dots n\}$ and define

$$S(\gamma)(t)^I = \int_{s=0}^t S(\gamma)(s)^{(i_1, \dots i_{k-1})} \gamma^* \mathrm{d}x_{i_k}$$

Iterated Integrals: Cochains on Path Spaces

Let $\gamma : I \to \mathbb{R}^n$ and $\{\mathrm{d}x_i\}_{i=1}^n$ the standard basis for the 1-forms on $\mathbb{R}^n$.

For $i \in \{1, \dots n\}$, define a new path, $S(\gamma)^i : I \to \mathbb{R}$ by

$$S(\gamma)(t)^i = \int_{s=0}^t \gamma^* \mathrm{d}x_i = \int_{s=0}^t \mathrm{d}\gamma_i = \gamma_i(t) - \gamma_i(0)$$

Iterate: Let $I = (i_1, i_2, \dots i_k)$, $i_\ell \in \{1, \dots n\}$ and define

$$S(\gamma)(t)^I = \int_{s=0}^t S(\gamma)(s)^{(i_1, \dots i_{k-1})} \gamma^* \mathrm{d}x_{i_k}$$

Write $S(\gamma)^I = S(\gamma)(1)^I \in \mathbb{R}$; so, $S(-)^I$ is a cochain on $P\mathbb{R}^n$.

Iterated Integrals: Cochains on Path Spaces

To see what's really going on, consider the evaluation map

$$\mathrm{ev} : \Delta^k \times P\mathbb{R}^n \to (\mathbb{R}^n)^k$$
$$\mathrm{ev}((t_1, \ldots, t_n), \gamma) = (\gamma(t_1), \ldots, \gamma(t_n))$$

Iterated Integrals: Cochains on Path Spaces

To see what's really going on, consider the evaluation map

$$\mathrm{ev} : \Delta^k \times P\mathbb{R}^n \to (\mathbb{R}^n)^k$$
$$\mathrm{ev}((t_1, \ldots, t_n), \gamma) = (\gamma(t_1), \ldots, \gamma(t_n))$$

On cochains, we have

$$C^*(\mathbb{R}^n)^{\otimes k} \xrightarrow{\mathrm{ev}^*} C^*(\Delta^k \times P\mathbb{R}^n) \xrightarrow{\int_{\Delta^k}} C^{*-k}(P\mathbb{R}^n)$$

and S(-)^I is the image of $\bigotimes_{\ell=1}^k \mathrm{d}x_{i_\ell}$ under this composition.

Iterated Integrals: Cochains on Path Spaces

Fun facts about $S(-)^I$, due to K.T. Chen:

Iterated Integrals: Cochains on Path Spaces

Fun facts about $S(-)^I$, due to K.T. Chen:

- invariant under reparamaterization and translation!

Iterated Integrals: Cochains on Path Spaces

Fun facts about $S(-)^I$, due to K.T. Chen:

- invariant under reparamaterization and translation!
- $S(-)^I \cdot S(-)^J = \sum_{K \in \mathrm{Sh}(I,J)} S(-)^K$

Iterated Integrals: Cochains on Path Spaces

Fun facts about $S(-)^I$, due to K.T. Chen:

- invariant under reparamaterization and translation!
- $S(-)^I \cdot S(-)^J = \sum_{K \in \mathrm{Sh}(I,J)} S(-)^K$

Definition: The *signature* of a path $\gamma : I \to \mathbb{R}^n$ is

$$S(\gamma) = (S(\gamma)^I \mid I = (i_1, \ldots i_k), k \geq 0)$$

- $\mathrm{S}(\gamma_1 * \gamma_2) = S(\gamma_1) \cdot S(\gamma_2)$

Iterated Integrals: Cochains on Path Spaces

Fun facts about $S(-)^I$, due to K.T. Chen:

- invariant under reparamaterization and translation!
- $$S(-)^I \cdot S(-)^J = \sum_{K \in \mathrm{Sh}(I,J)} S(-)^K$$

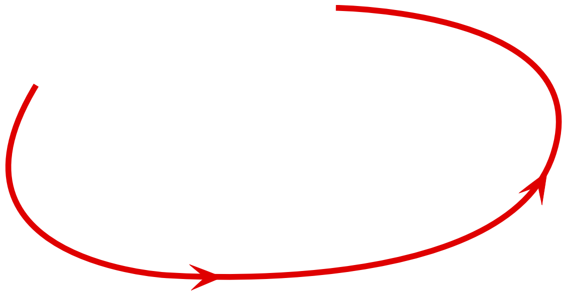
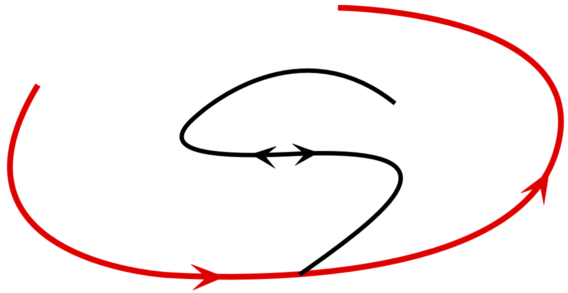
Definition: The *signature* of a path $\gamma : I \to \mathbb{R}^n$ is

$$S(\gamma) = (S(\gamma)^I \mid I = (i_1, \ldots i_k), k \geq 0)$$

- $\mathrm{S}(\gamma_1 * \gamma_2) = S(\gamma_1) \cdot S(\gamma_2)$
- $S(\bar{\gamma}) = S(\gamma)^{-1}$

Iterated Integrals: Cochains on Path Spaces

Theorem (Chen, Hambly & Lyons):
$S(-)$ is a complete invariant of bounded variation paths up to reparameterization, translation and ”tree-like equivalence”.

$S($  $) = S($  $)$

Level 2 Signatures

Level 2 Signatures

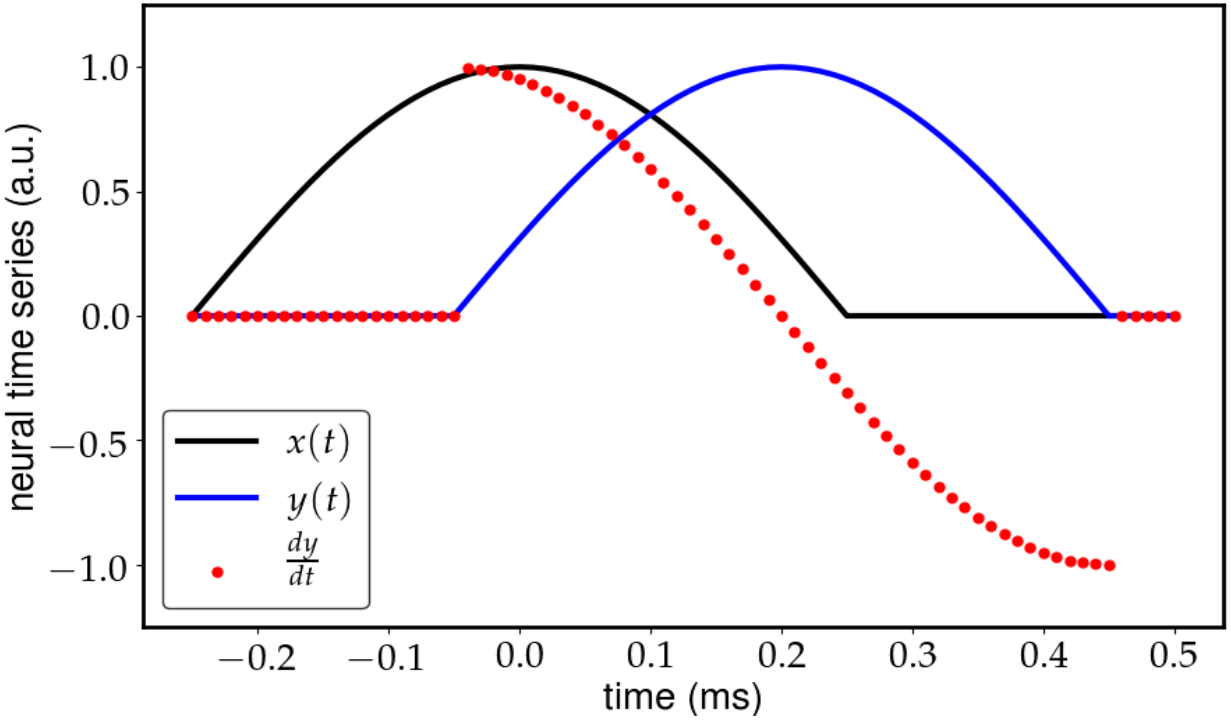
Suppose $\gamma(0) = \vec{0}$.

$$
\begin{aligned}
S(\gamma)^{i,j} &= \int_{t=0}^{1} \int_{s=0}^{t} \mathrm{d}\gamma_i \; \mathrm{d}\gamma_j \\
&= \int_{t=0}^{1} \gamma_i(t) \mathrm{d}\gamma_j \\
&= \int_{t=0}^{1} \gamma_i(t) \gamma_j'(t) \mathrm{d}t
\end{aligned}
$$

Level 2 Signatures

Suppose $\gamma(0) = \vec{0}$.

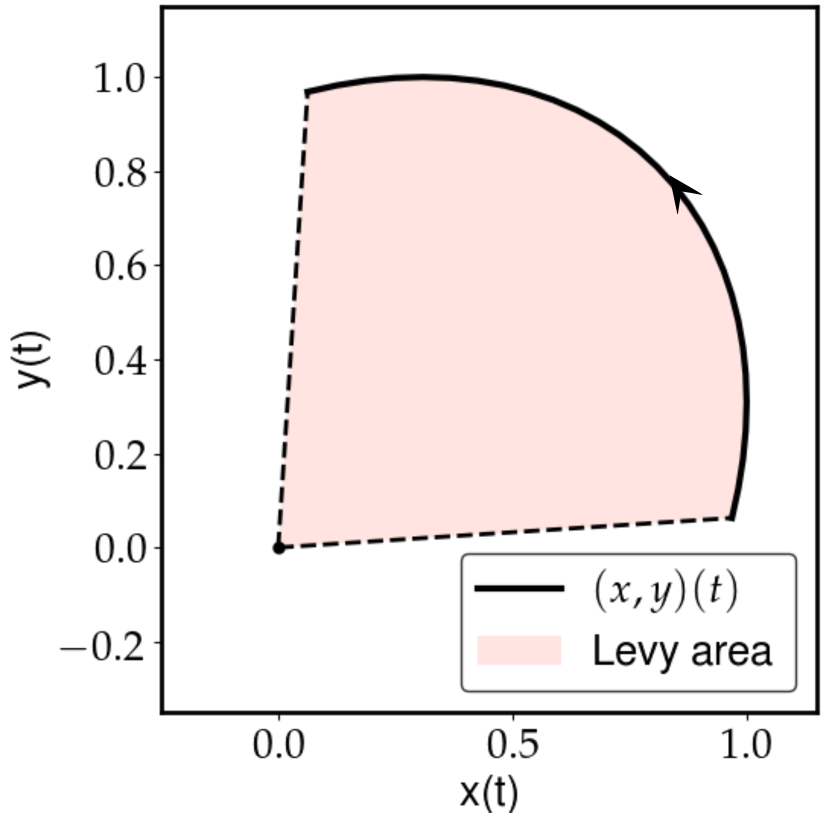
$$S(\gamma)^{i,j} = \int_{t=0}^{1} \int_{s=0}^{t} \mathrm{d}\gamma_i \, \mathrm{d}\gamma_j$$
$$= \int_{t=0}^{1} \gamma_i(t) \mathrm{d}\gamma_j$$
$$= \int_{t=0}^{1} \gamma_i(t) \gamma_j'(t) \mathrm{d}t$$



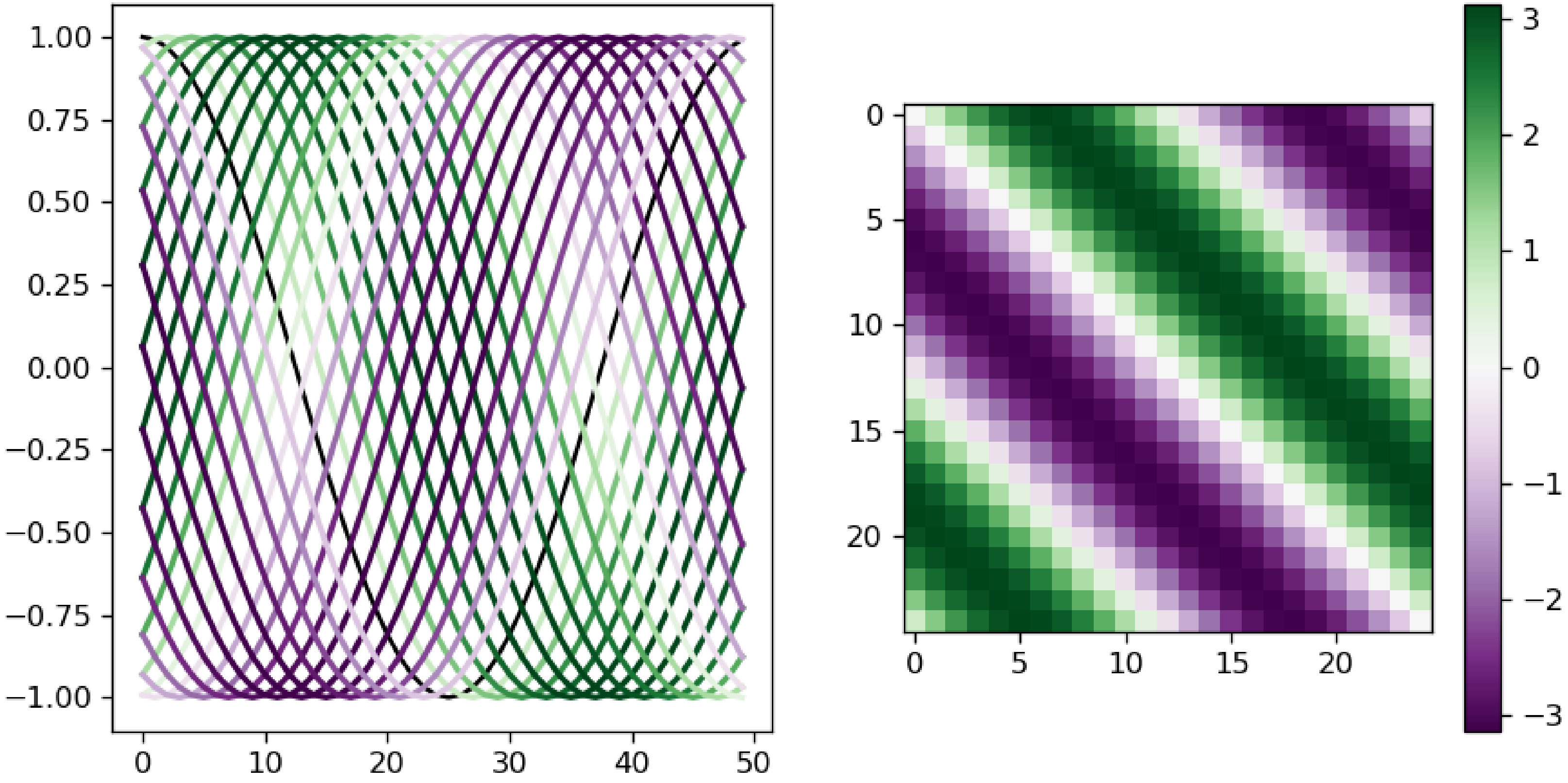
Lévy Area

Definition: The *Lévy area* of a (piecewise smooth) planar path $\gamma : I \to \mathbb{R}^2$ is

$$LA(\gamma) = \frac{1}{2}(S(\gamma)^{1,2} - S(\gamma)^{2,1})$$

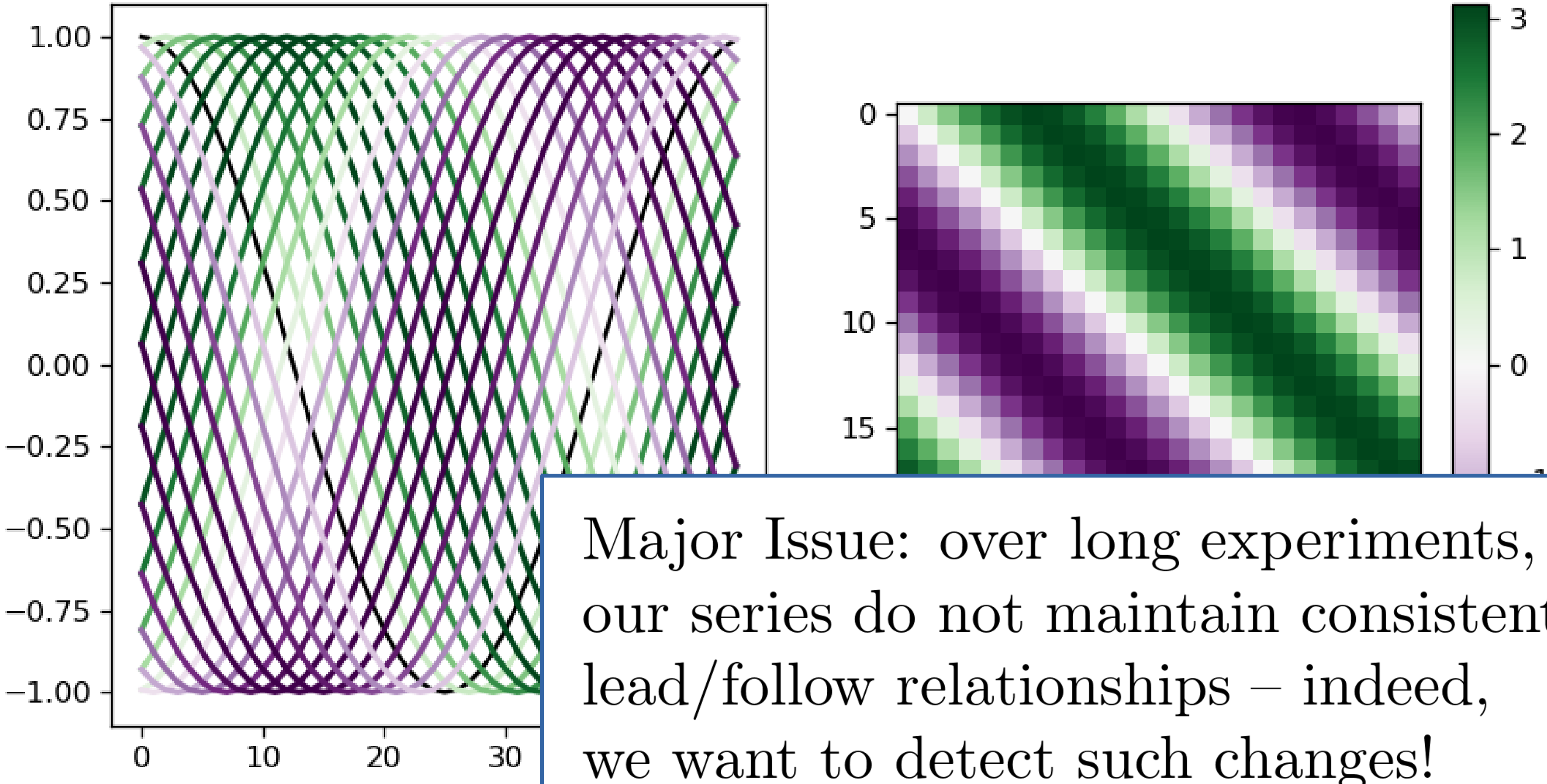


Lévy Area



1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00
0
10
20
30
40
50
0
5
10
15
20
0
5
10
15
20
3
2
1
0
−1
−2
−3

Lévy Area



Major Issue: over long experiments, our series do not maintain consistent lead/follow relationships – indeed, we want to detect such changes!

Lévy Area Lead Coefficient

The *rate of change* of the Levy area,

$$LA'_{ij}(\gamma)(t) = \frac{1}{2}(\gamma_i(t)\gamma'_j(t) - \gamma_j(t)\gamma'_i(t))$$

is a temporally local measure of the lead/follow relationship between pairs of time series.

Lévy Area Lead Coefficient

The *rate of change* of the Levy area,

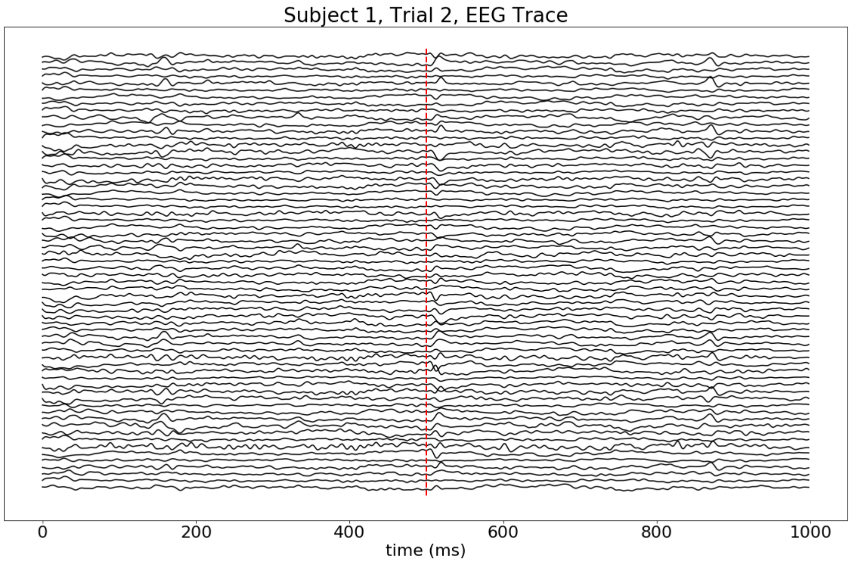
$$LA'_{ij}(\gamma)(t) = \frac{1}{2}(\gamma_i(t)\gamma'_j(t) - \gamma_j(t)\gamma'_i(t))$$

is a temporally local measure of the lead/follow relationship between pairs of time series.

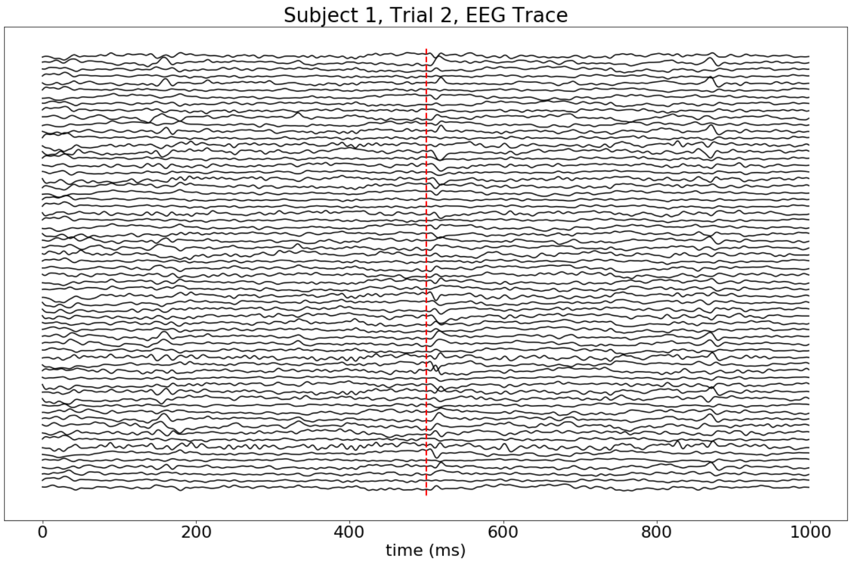
To apply this to data

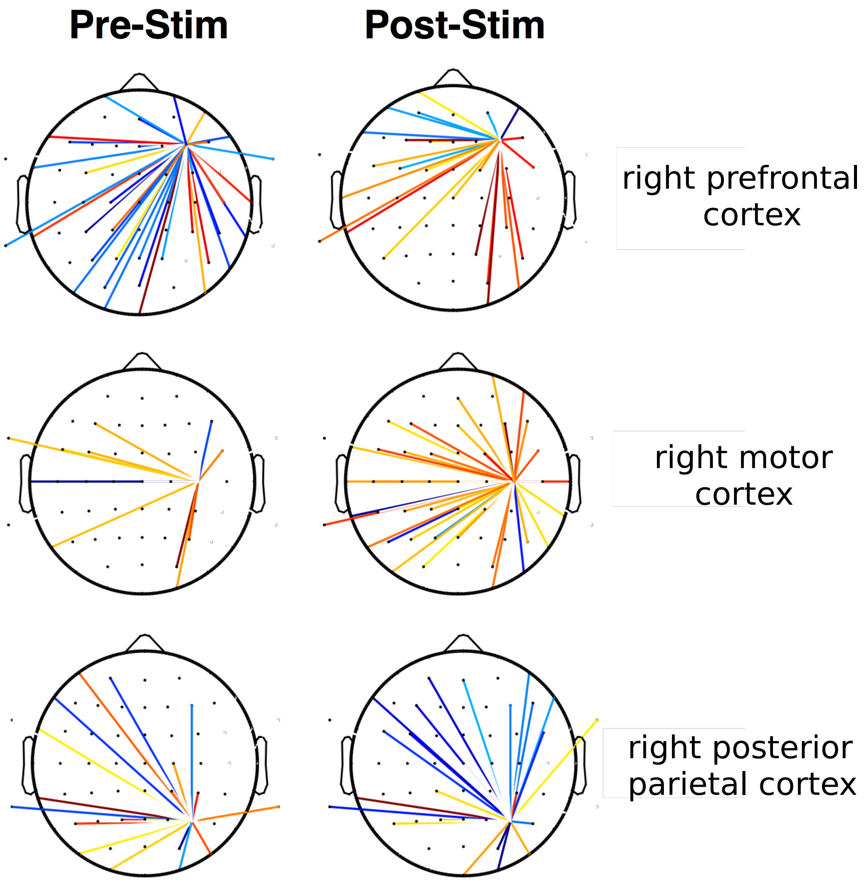
1. we compute $LA'_{ij}(\gamma)(t)$ for all pairs of time series;
2. we compute $LA'_{ij}(\gamma)(t)$ for time-shuffled controls;
3. we threshold each $LA'_{ij}(\gamma)(t)$, retaining only 3σ-significant epochs of LA' lasting more than 10 ms, and
4. define the *Levy Area Lead Coefficient* to be the mean of these thresholded time series.

What do we see in the data?



What do we see in the data?





What's Next?

These data aren't *really* a time series in $\mathbb{R}^{32}$.
They're a time series in $F(D^2, \mathbb{R})$.

However, this iterated integral methodology is highly flexible.

What's Next?

These data aren't *really* a time series in $\mathbb{R}^{32}$.
They're a time series in $F(D^2, \mathbb{R})$.

However, this iterated integral methodology is highly flexible.

Theorem: (Andserson spectral sequence)
Let X be a simplicial finite set w.h.e. to a finite simplicial set,
Y a topological space so that $dim(X) \leq conn(Y)$.
Then there is a spectral sequence

$$\{E^1_{*,q} = \bigotimes_{\sigma \in X_q} H_*(Y)\} \Rightarrow H_*(Y^{|X|}; \mathbb{R})$$

What's Next?

By (Patras and Thomas ,03), this is computable via the evaluation map!

$$\text{ev} : \Delta^k \times ||Y^X|| \to Y^X[k]$$

On cochains, we have

$$C^*(Y^X[k]) = C^*(\prod_{\sigma \in X_k} Y) \xrightarrow{\text{ev}^*} C^*(\Delta^k \times ||Y^X||) \xrightarrow{\int_{\Delta^k}} C^{*-k}(||Y^X||)$$

And any "sufficiently nice" cochain model for Y will work here.

Thanks!