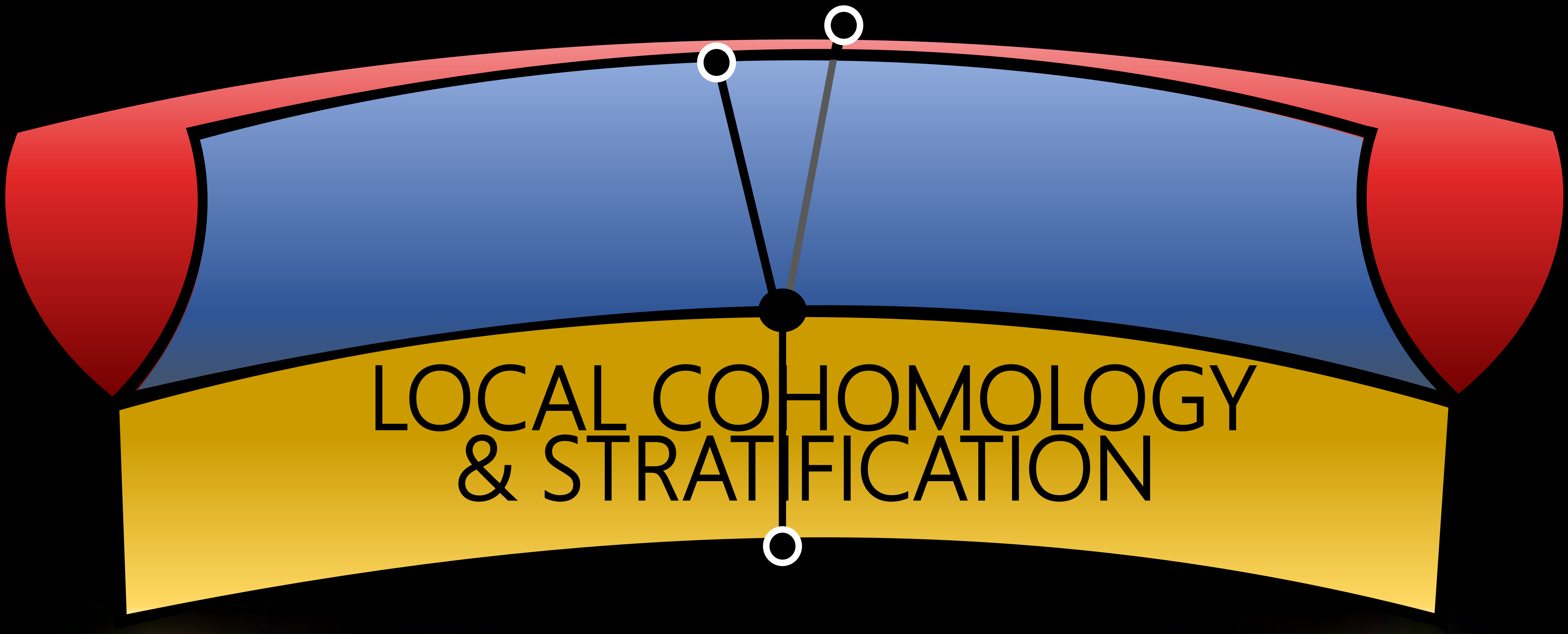
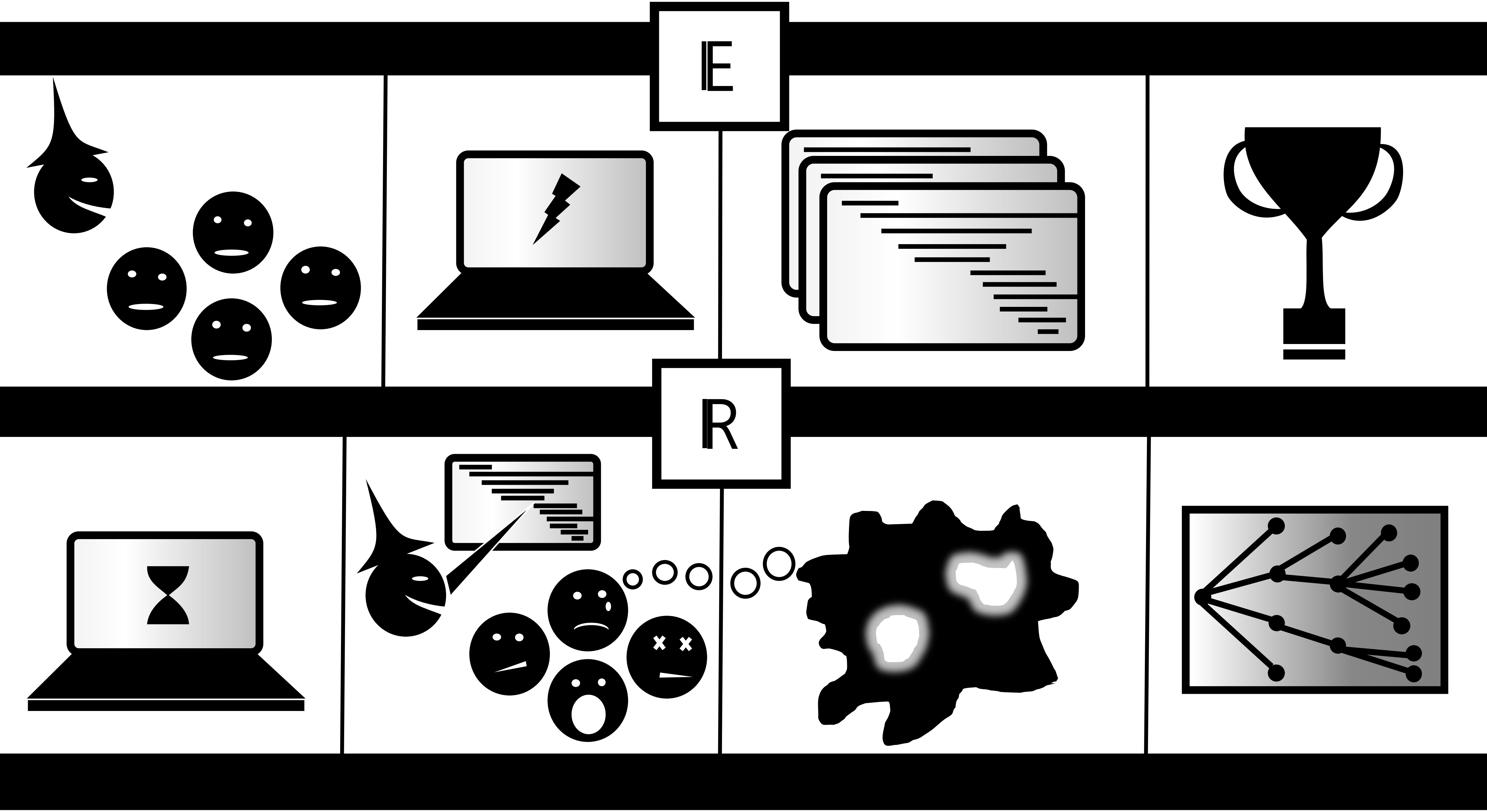


LOCAL COHOMOLOGY & STRATIFICATION

VIDIT NANDA

OXFORD | TURING | IAS

5TH JUNE 2018 @ THE ABEL SYMPOSIUM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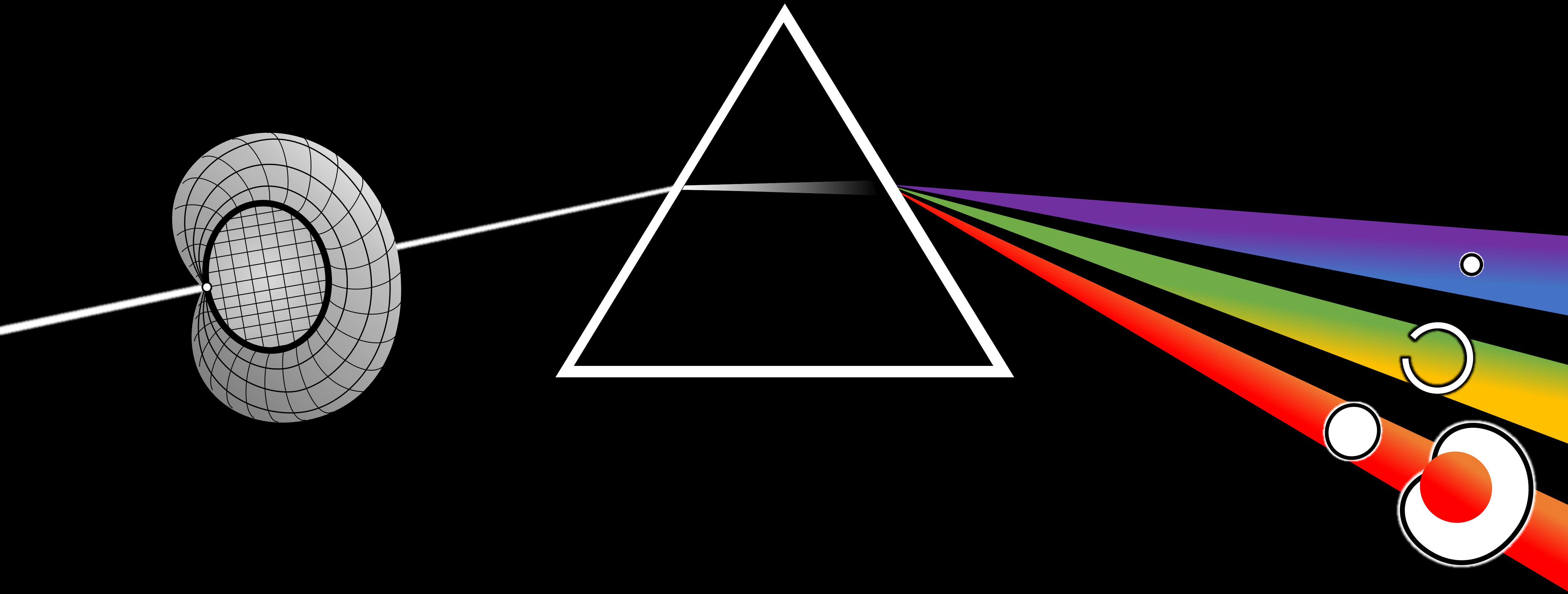


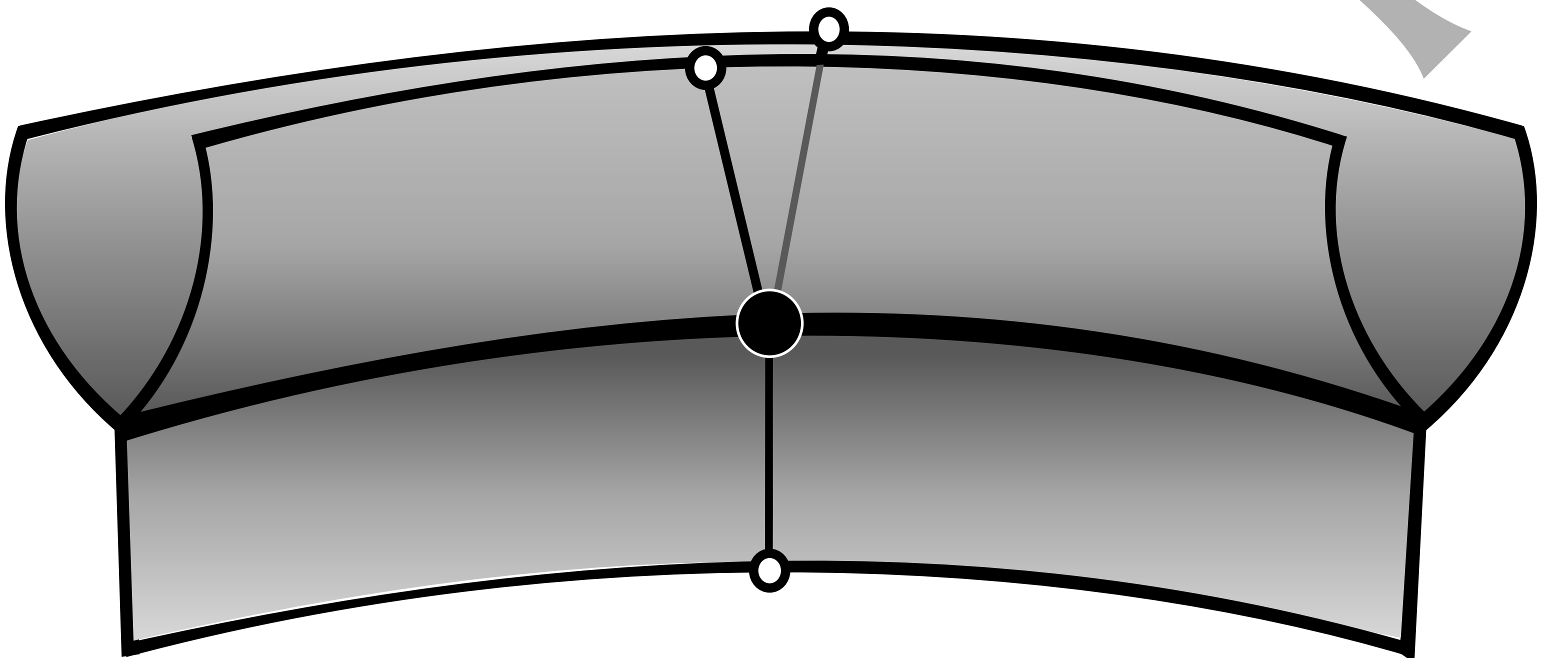
STRATIFICATION

An (n-dimensional, cohomological) **stratification** of a nice space X is a filtration by closed subspaces

$$\varnothing = X_{-1} \subset X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X, \text{ where}$$

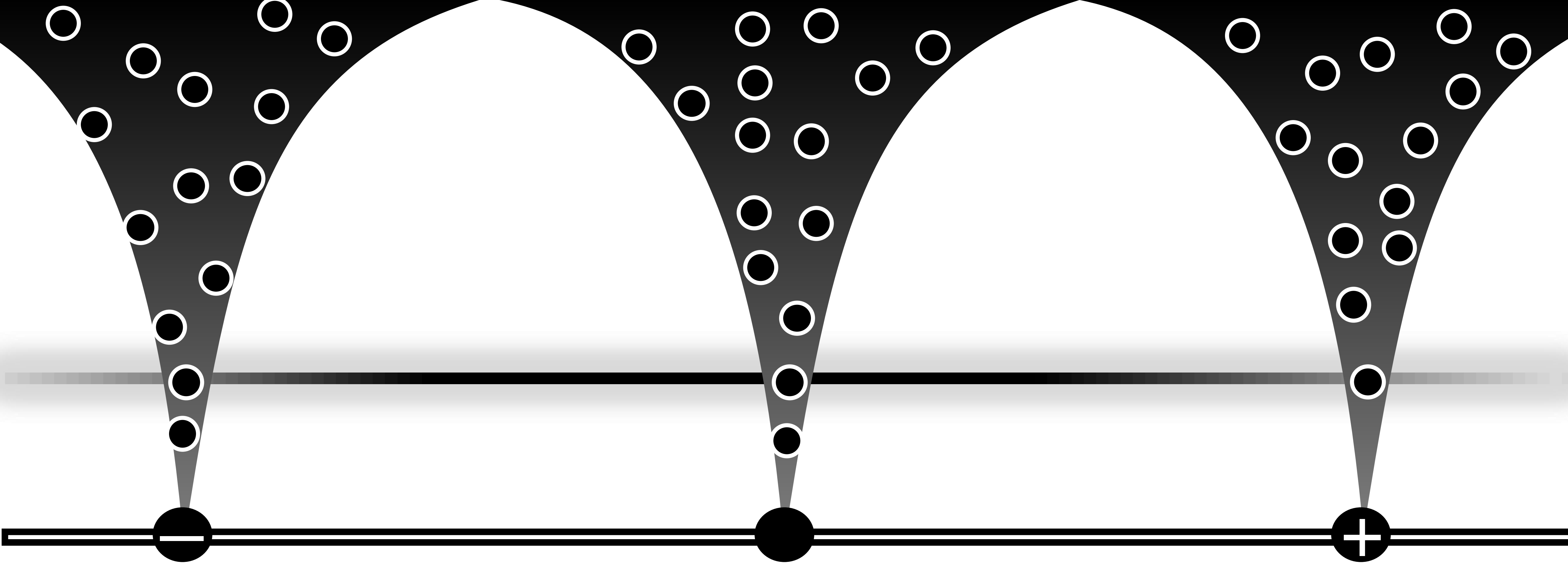
each connected component σ of $X_d - X_{d-1}$, called a *d-stratum*, obeys two rules.

If $\sigma \cap \bar{\tau} \neq \varnothing$ for another stratum τ, then $\sigma \subset \bar{\tau}$ and $\dim \tau \geq d$. This is a partial order on the set of strata.	There is an $(n-d-1)$-dim stratified space $L = L(\sigma)$ so that each $p \in \sigma$ admits an open U so that: $(U \cap X_i) \simeq (\mathrm{Cone}(L_{i-d-1}) \times \mathbb{R}^d)$ for $d \leq i \leq n$.
F	L



GOAL: COARSEN A GIVEN CELLULAR STRATIFICATION TO THE CANONICAL ONE.

QUESTION: HOW?



PERSISTENT IDEA:

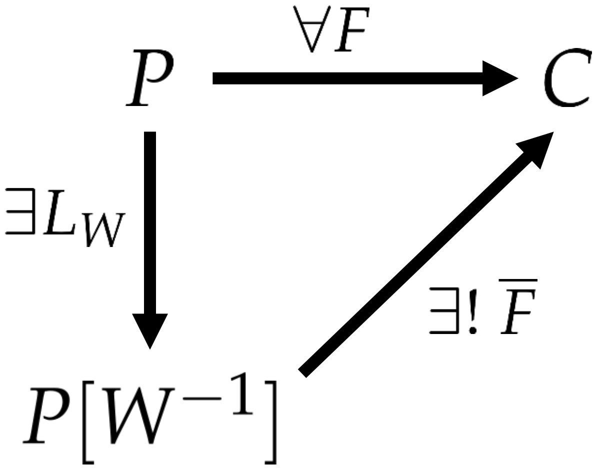
LOCALIZE THE POSET OF CELLS IN A CW COMPLEX ABOUT SOMETHING NATURAL

Let $(P, >)$ be a poset, and let $W = \{(x_\bullet \geq y_\bullet)\}$ be a subset of $P \times P$ that is *closed*: meaning, $(x \geq y)$ in W and $(y \geq z)$ in W forces $(x \geq z)$ in W.

The **localization** of P about W is the optimal category $P[W^{-1}]$ which contains a copy of P where all relations in W have become isomorphisms.

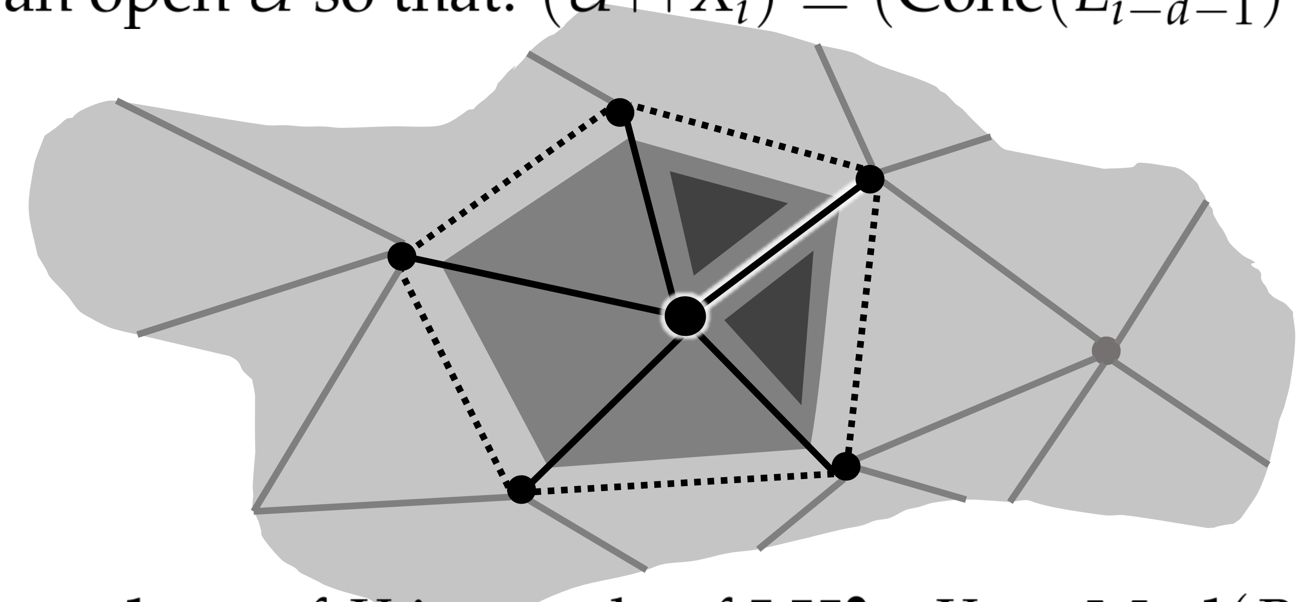
It has the same objects as P, but its morphisms are (equivalence classes of) *W-zigzags*:

$$z \geq y_1 \leq x_1 \geq \cdots \geq y_k \leq x_k \geq z'$$



WHAT W RELATES CELLS IN THE SAME STRATUM?

L There is an $(n-d-1)$-dim stratified space $L = L(\sigma)$ so that each $p \in \sigma$ admits an open U so that: $(U \cap X_i) \simeq (\text{Cone}(L_{i-d-1}) \times \mathbb{R}^d)$ for $d \leq i \leq n$.

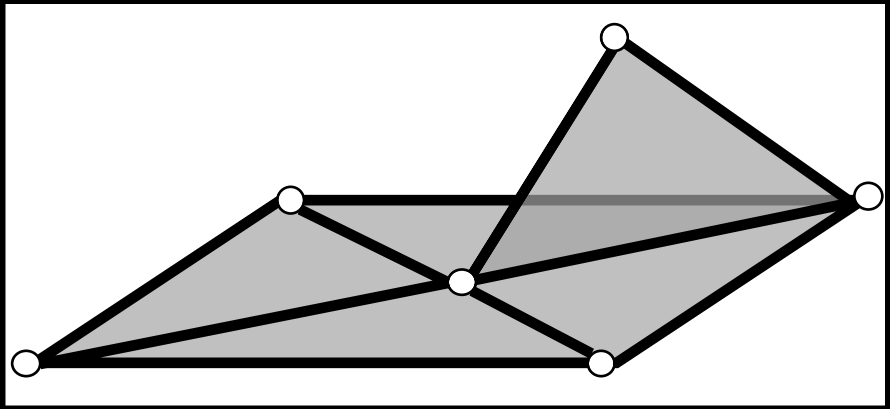


The **local cohomology** of X is a cosheaf $\mathbf{LH}^\bullet : X \to \mathbf{Mod}(R)$ whose stalk over the cell x is

$$\mathbf{LH}^\bullet(x) = H^\bullet\left(C_x^0 \xrightarrow{d_x^0} C_x^1 \xrightarrow{d_x^1} C_x^2 \xrightarrow{d_x^2} C_x^3 \xrightarrow{d_x^3} \cdots ; R\right)$$

(here C_x^i is generated by all cells z of dimension i with x as a face, and d_x^i is obtained by restricting the d^i matrix which computes the cohomology of X)

GUESS: $(x > y)$ is in W iff $\text{LH}(x) = \text{LH}(y)$



The central vertex has the same local cohomology as all the simplices in its star *except* the trivalent edge. By the frontier axiom, it can not lie on any 2-stratum.

UPWARD CLOSURE

Set $E_0 = \{(x \geq y) \mid \mathbf{LH}^\bullet(x) \simeq \mathbf{LH}^\bullet(y)\}$.

$W_0 = \{(x \geq y) \in E_0 \mid \mathbf{LH}^\bullet(y) \simeq \tilde{H}^\bullet(\mathbb{S}^n)$ and $(z \geq y) \in E_0$ for each $z \geq y\}$.

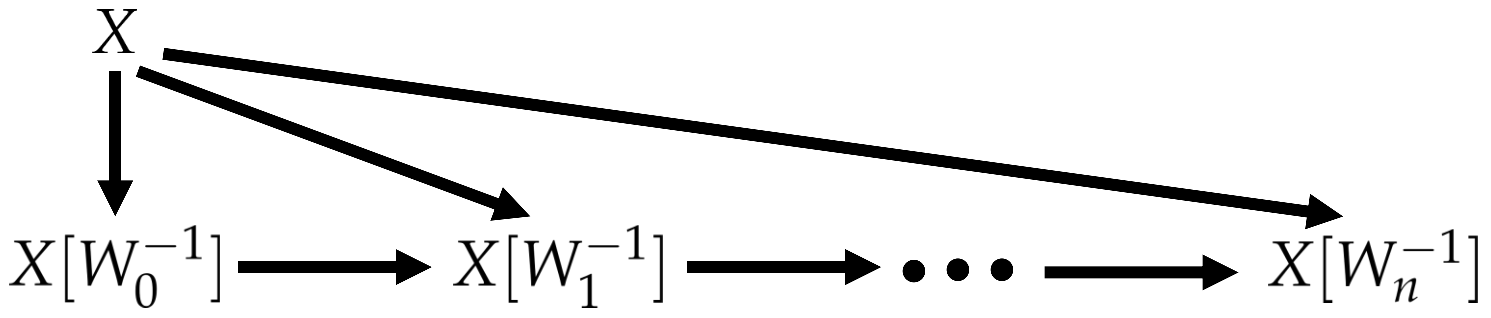
THM

[N, 2017] If X is an n-dimensional regular CW complex, then its canonical n-strata are isomorphism classes of n-cells in $X[W_0^{-1}]$

LOWER STRATA:

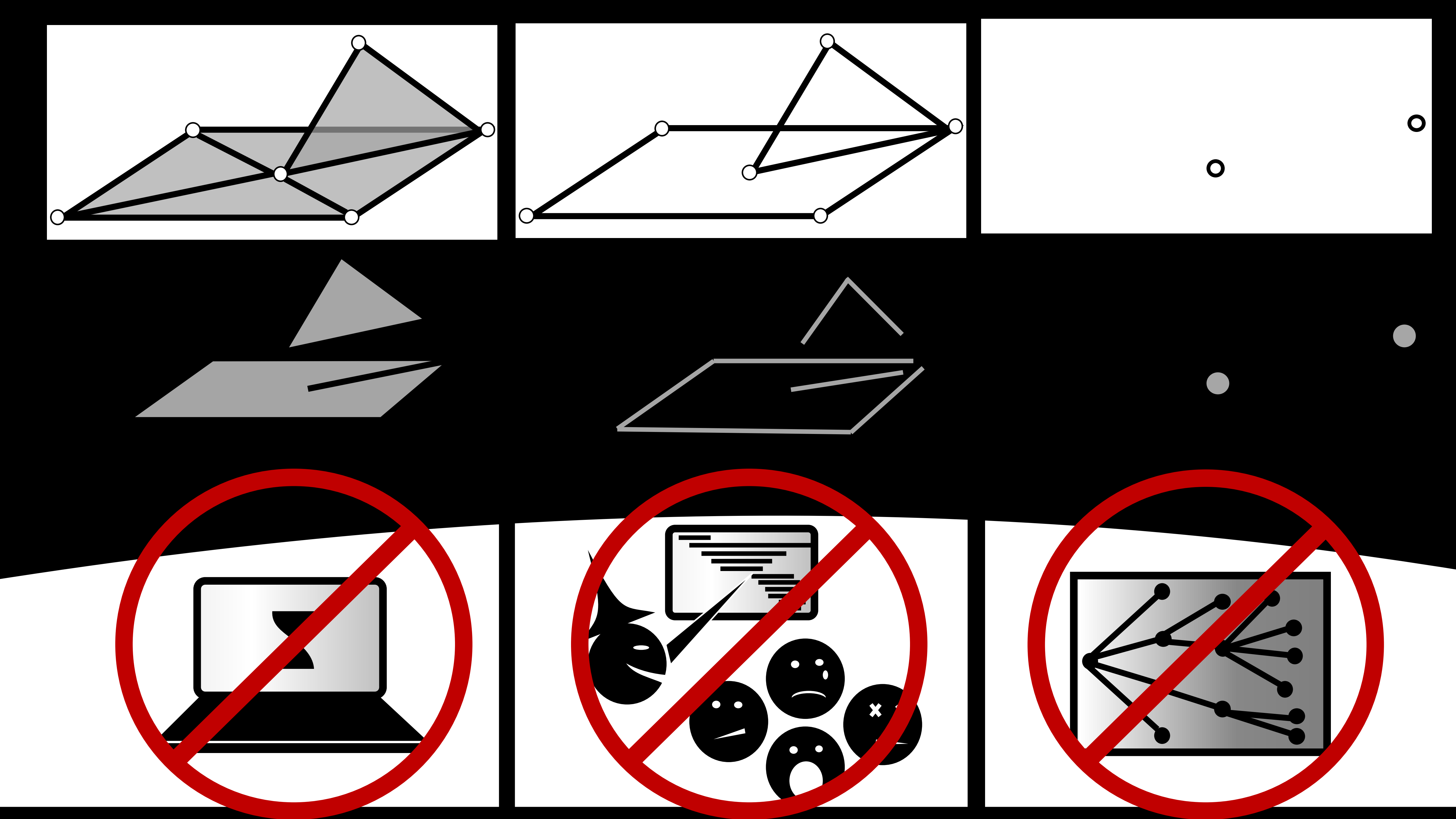
Given a triple (Y_0, E_0, W_0) containing our cell complex $Y_0 = X$ and the two sets $W_0 \subset E_0 \subset X \times X$,

Set $Y_1 = Y_0 - \{\text{cells in n}-\textit{strata}\}$, this has dim $\leq (n-1)$
Set $E_1 = E_0 \cap \{\text{blah blah local cohomology equivalences of } Y_1\}$
Set $W_1 = W_0 \cup \{\text{blah blah upward closure and homology of } \mathbb{S}^{n-1}\}$
And repeat these steps with the triple (Y_1, E_1, W_1)



THM

[N, 2017] The canonical $(n-d)$ strata of X correspond bijectively with the isomorphism classes of $(n-d)$ cells from Y_d in $X[W_k^{-1}]$ for *every* $k \geq d$.



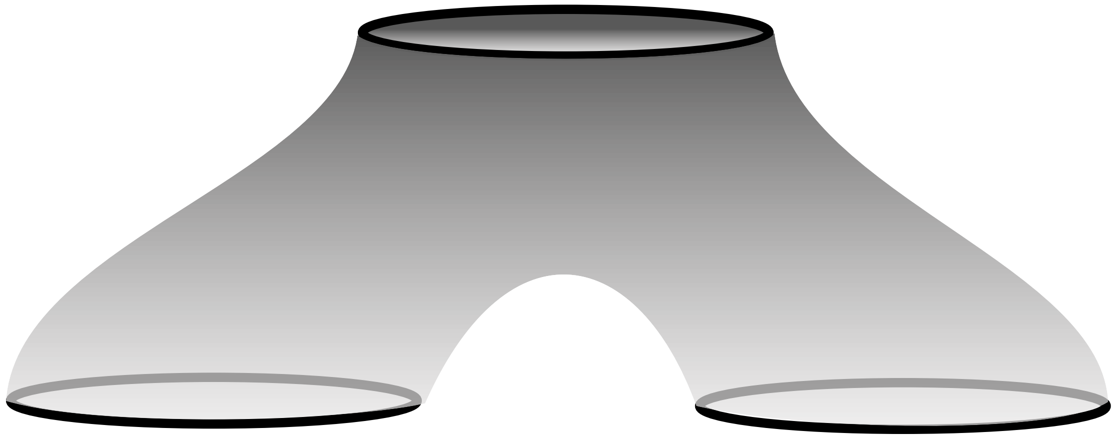
THE GHOST OF LOCALIZATION PAST

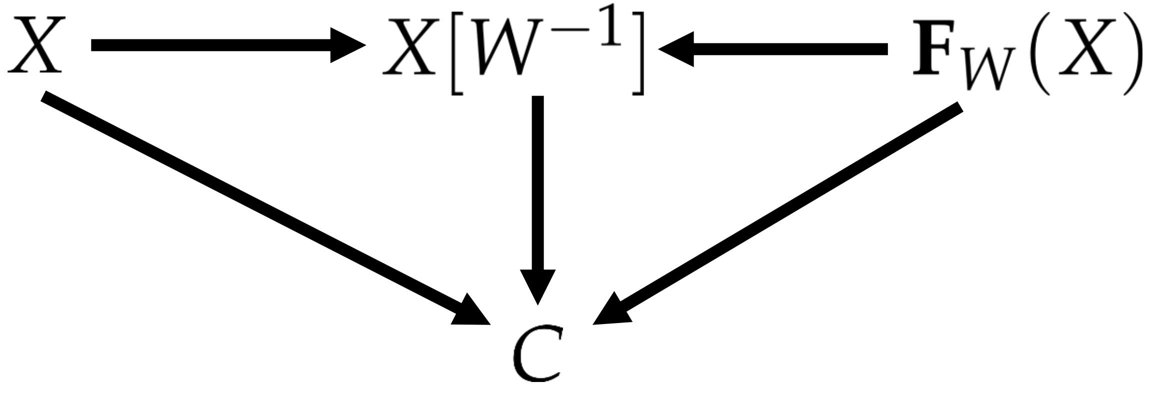
Let X be a finite regular CW complex, and let W be the collection of cell pairs $(x_\bullet > y_\bullet)$ associated to a discrete Morse function on X.

THM

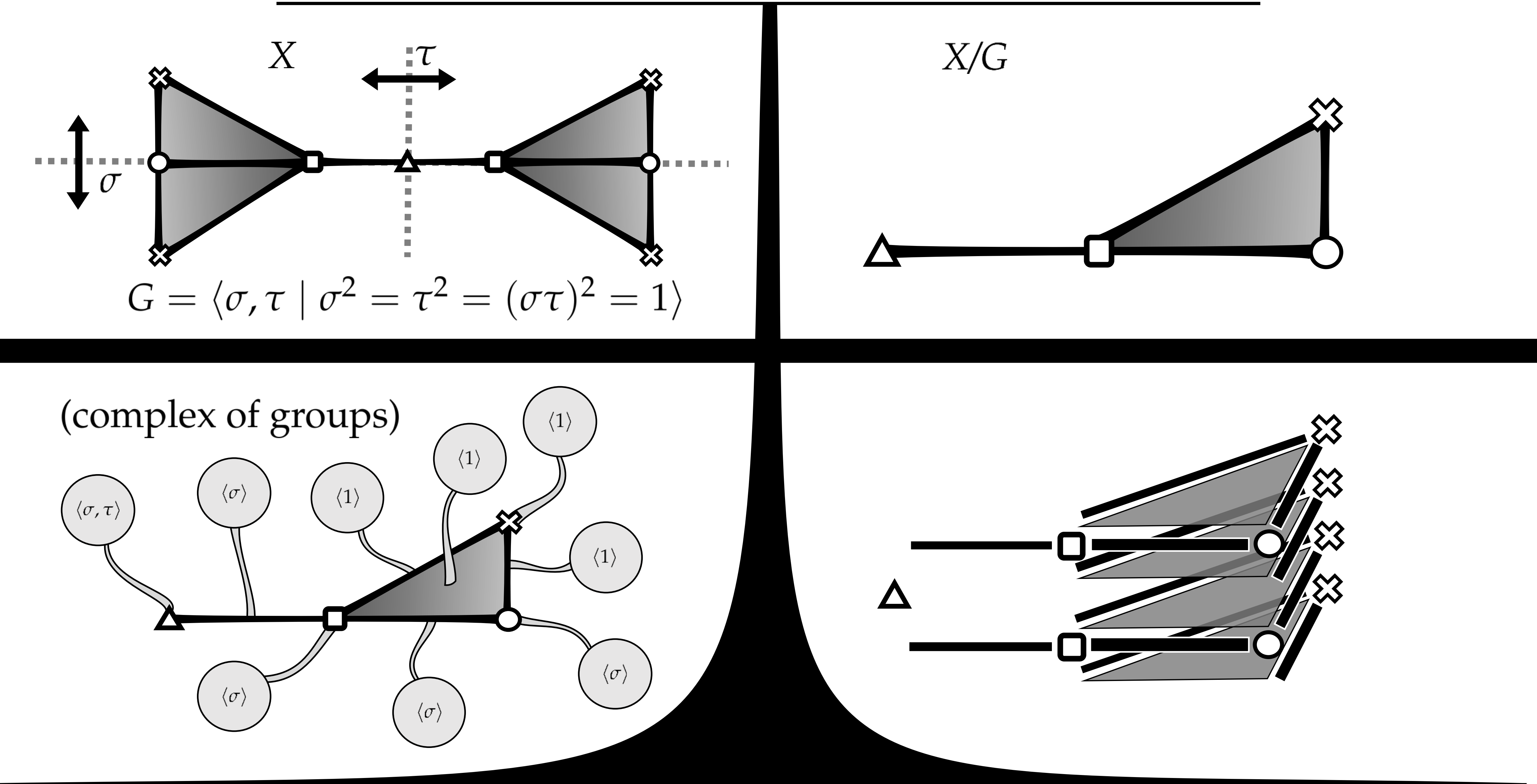
[N, 2015] There is a category $\mathbf{F}_W(X)$ whose objects are the critical (unpaired) cells, whose morphisms are W-localized zigzags, and whose classifying space is homotopy-equivalent to X.

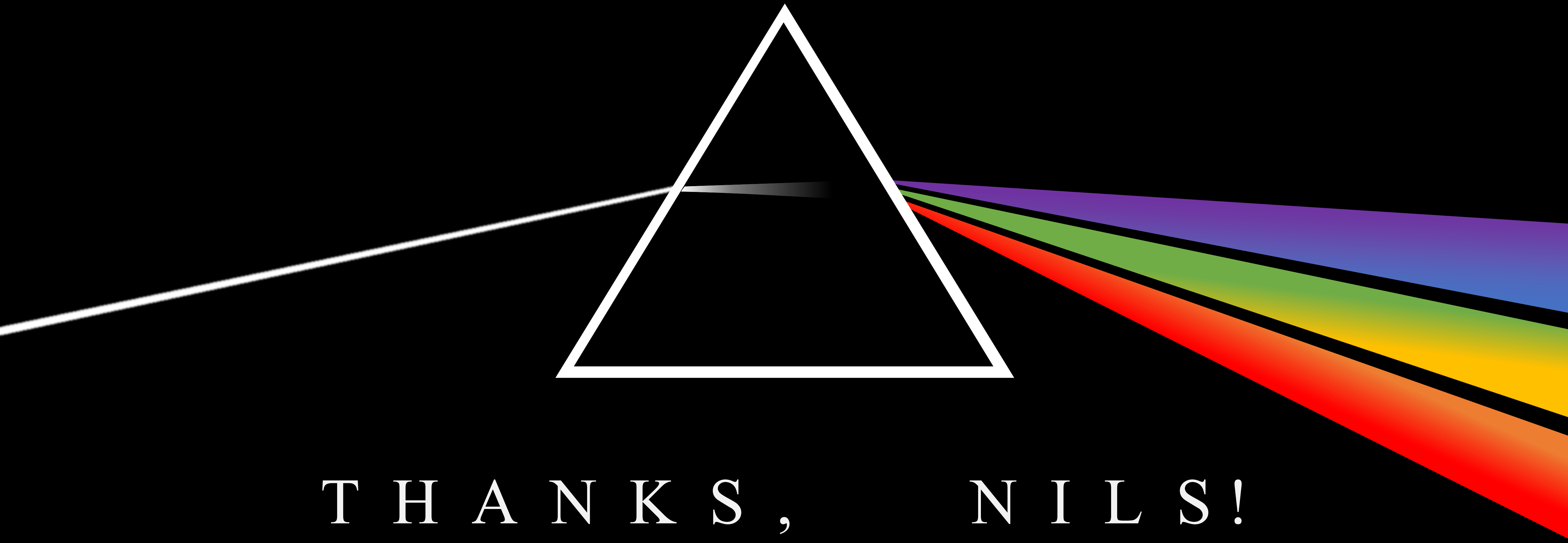
(Both functors $X \longrightarrow X[W^{-1}] \longleftarrow \mathbf{F}_W(X)$ induce homotopy equivalences.)





THE GHOST OF LOCALIZATION FUTURE





THANKS, NILS!

N, *Discrete Morse theory and localization* (JPAA 2018)
N, *Local cohomology and stratification* (arXiv 2017)
Carbone, N and Naqvi, *Equivariant simplicial reconstruction* (coming soon)