Persistent Local Systems

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joint work with

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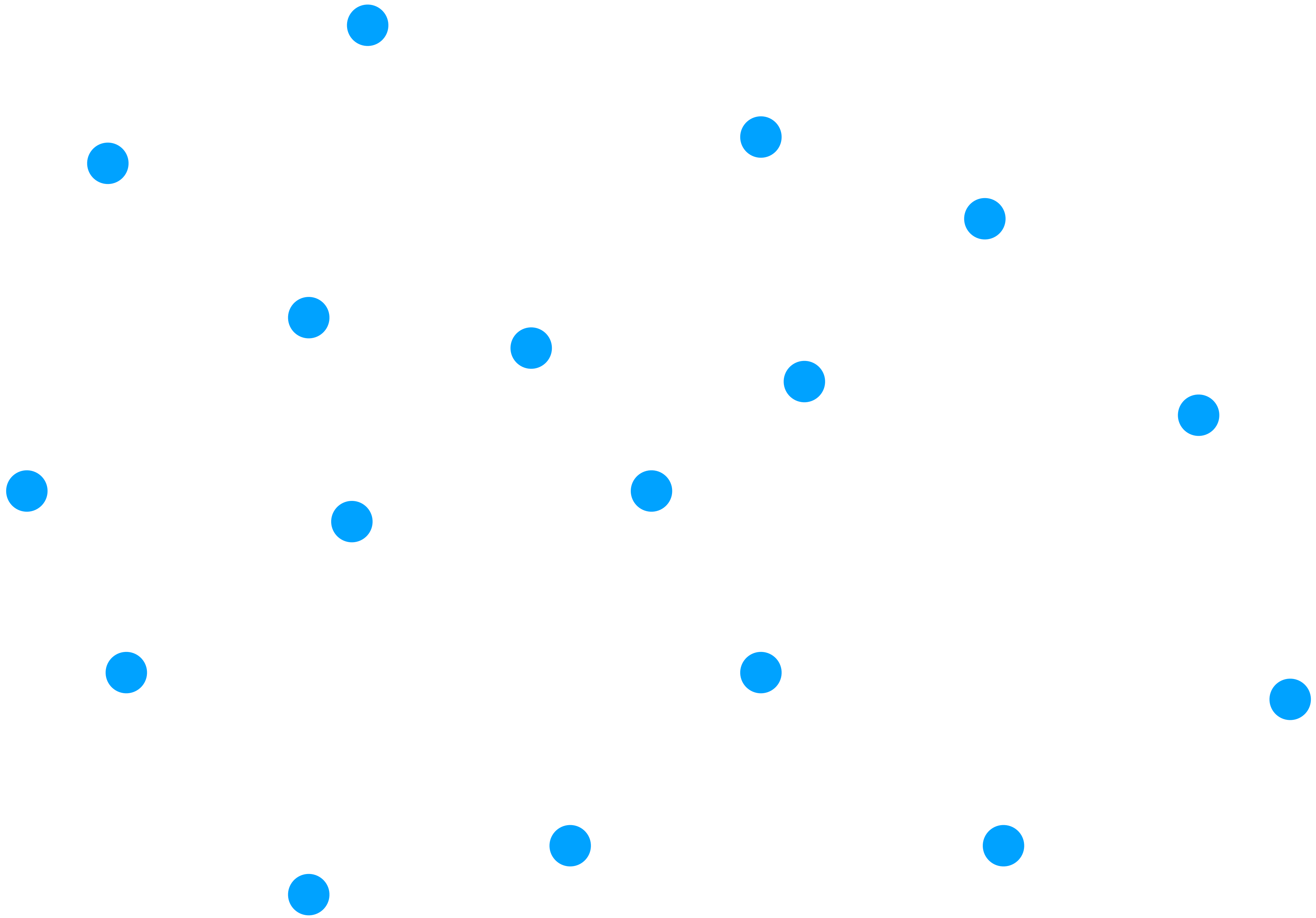
- Lays *foundation* for a theory of persistence in the setting of constructible maps to manifolds

$$\begin{array}{c} X \\ \big\downarrow f \\ M \end{array}$$

- Real Algebraic Maps
- Real Analytic Maps
- Proper Piecewise Linear Maps
- Structurally Stable Smooth Maps

Involves new (co)sheaf theoretic methods

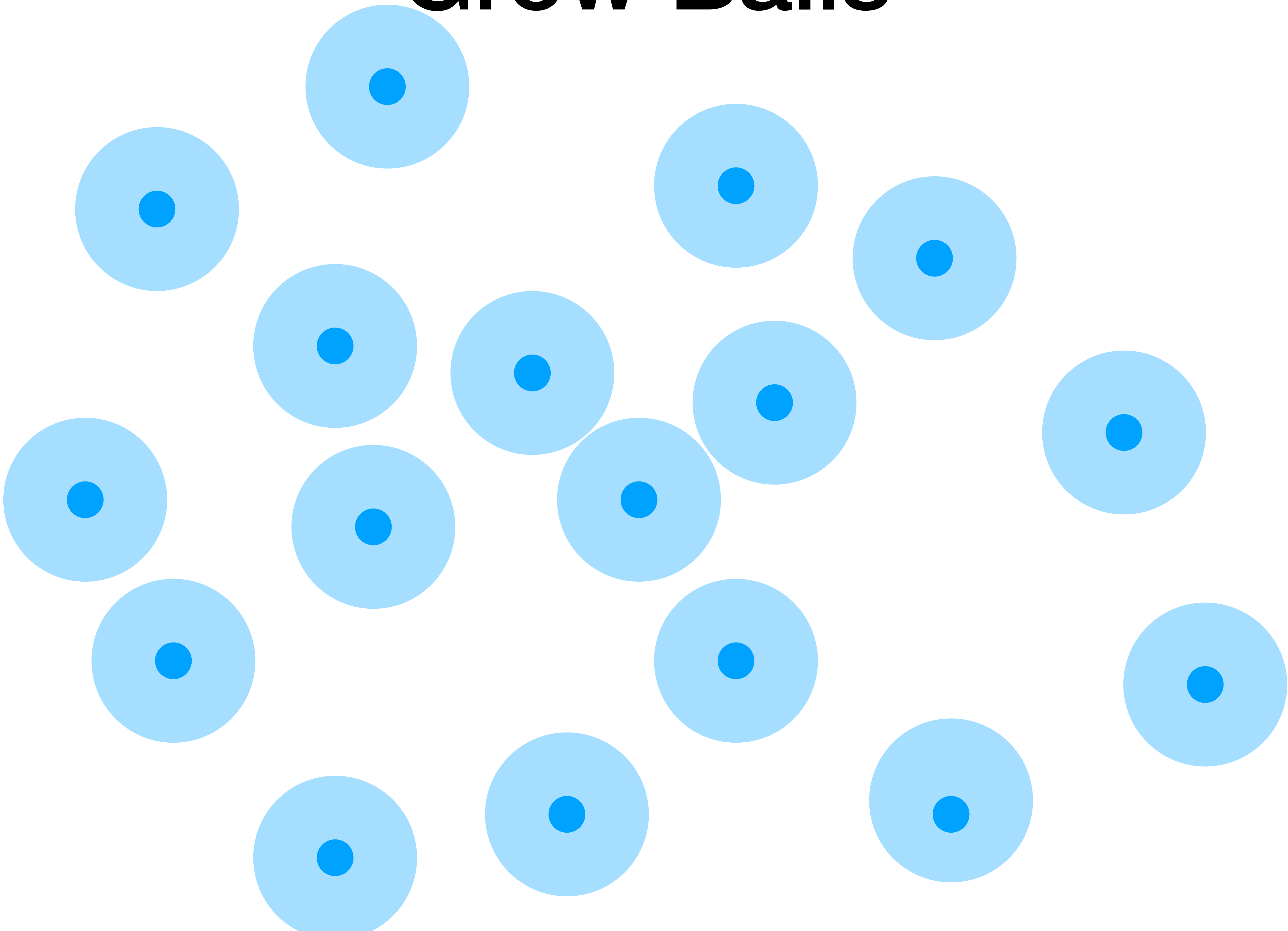
Point Cloud



$$P \subseteq \mathbb{R}^n$$

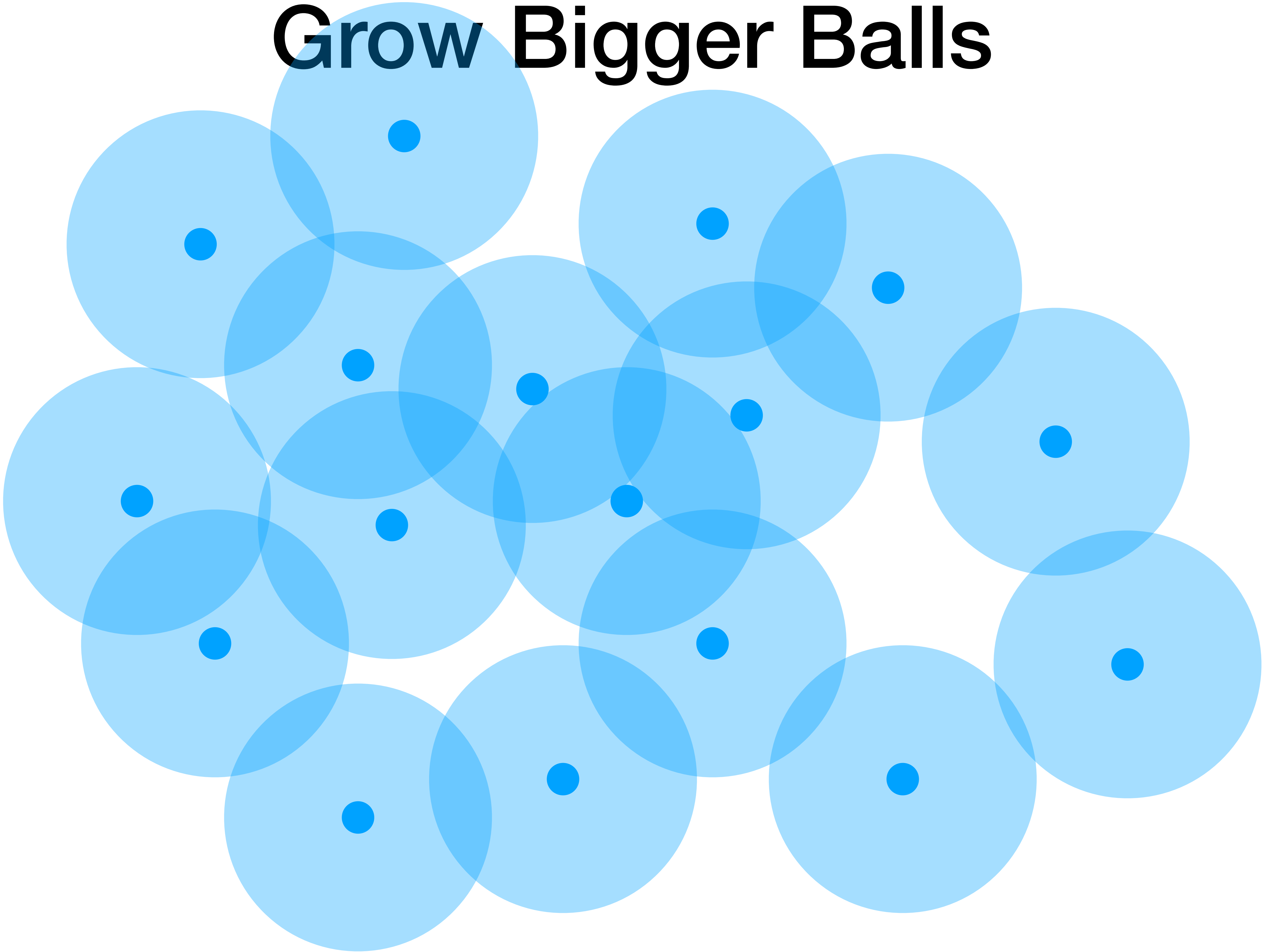
What is the shape of this point cloud at all scales?

Grow Balls



$$X(r) \equiv \bigcup_{p \in P} B_p(r)$$

Grow Bigger Balls



$$X(r) \equiv \bigcup_{p \in P} B_p(r)$$

Filtration

$$X(0) \hookrightarrow X(r_1) \hookrightarrow X(r_2) \hookrightarrow \cdots \hookrightarrow X(\infty) = \mathbb{R}^n$$

$$\mathsf{H}X(0) \to \mathsf{H}X(r_1) \to \mathsf{H}X(r_2) \to \cdots \to \mathsf{H}X(\infty)$$

Using coefficients in a field k

$$\mathsf{F} : (\mathbb{R}, \leq) \to \mathsf{Vect}_{\mathsf{k}}$$

What Persists?

$$\mathsf{F} : (\mathbb{R}, \leq) \to \mathsf{Vect}_{\mathsf{k}}$$

Given two values $r \leq s$ what persists along the entire interval $[r, s]$?

Answer: the **persistent homology group** which is the image of the map $\mathsf{F}(r \leq s)$

Edelsbrunner, Letcher, Zomorodian. *Topological persistence and simplification*. 2002

Property I of the PH Group

$$\mathsf{F} : (\mathbb{R}, \leq) \to \mathsf{Vect_k}$$

Consider two intervals of the real line: $[p, s] \supseteq [q, r]$

$$\begin{array}{ccc} \mathsf{F}(p) & \longrightarrow & \mathsf{F}(q) \\ \downarrow & & \downarrow \\ \mathsf{F}(s) & \longleftarrow & \mathsf{F}(r) \end{array}$$

$$\mathrm{rank}\ \mathsf{F}(p \leq s) \leq \mathrm{rank}\ \mathsf{F}(q \leq r)$$

Property I of the PH Group

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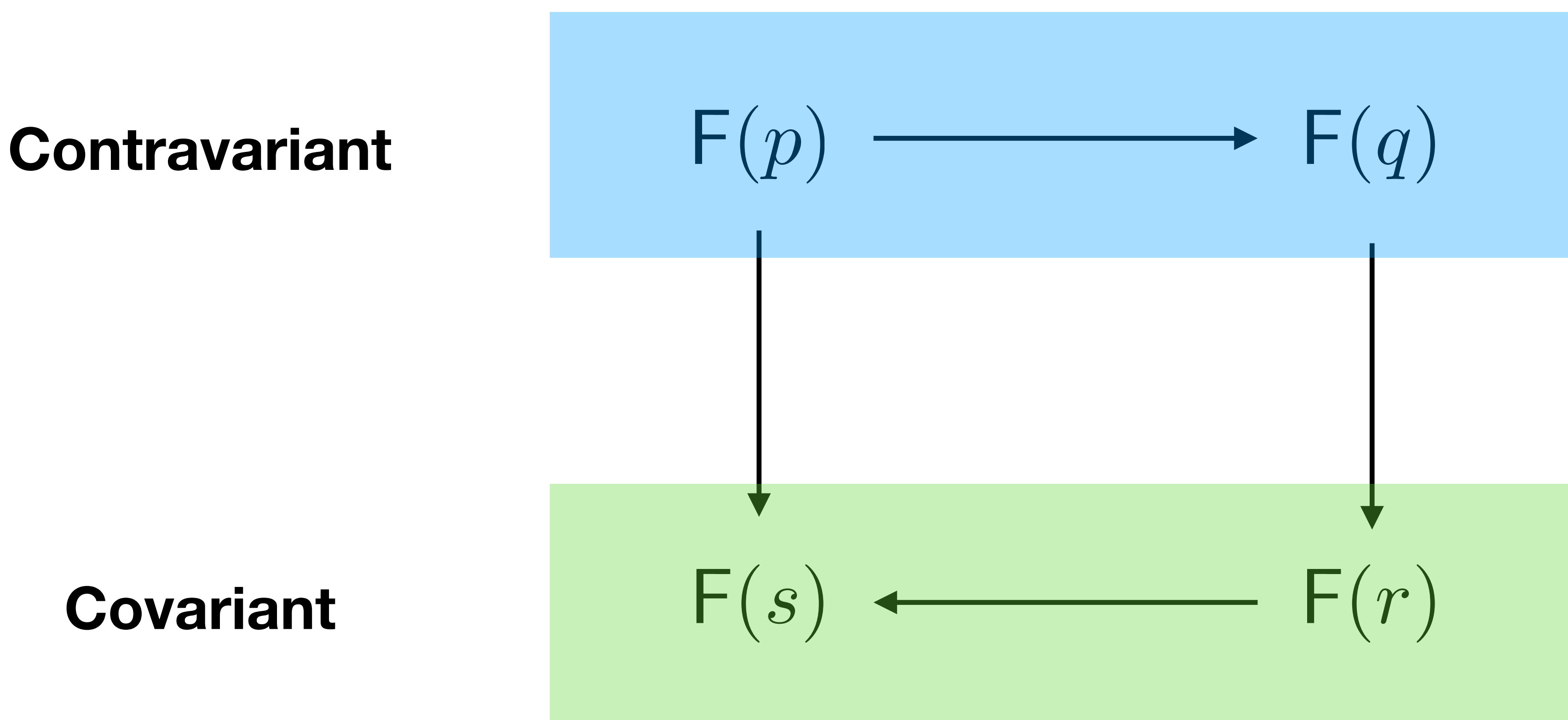


$$\text{rank } \mathsf{F}(p \leq s) \leq \text{rank } \mathsf{F}(q \leq r)$$

Property II of the PH Group

$\mathsf{F}, \mathsf{G} : (\mathbb{R}, \leq) \to \mathsf{Vect_k}$ ϵ - interleaved

For every long enough interval $[p, q]$

$$\begin{array}{ccc} \mathsf{F}(p) & \longrightarrow & \mathsf{G}(p+\epsilon) \\ \downarrow & & \downarrow \\ \mathsf{F}(q) & \longleftarrow & \mathsf{G}(q-\epsilon) \end{array}$$

$$\text{rank } \mathsf{F}(p \leq q) \leq \text{rank } \mathsf{G}(p+\epsilon \leq q-\epsilon)$$

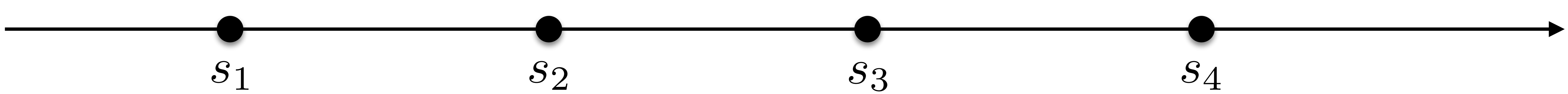
Constructible Persistence Modules

The persistence diagram can be defined solely from persistent homology groups

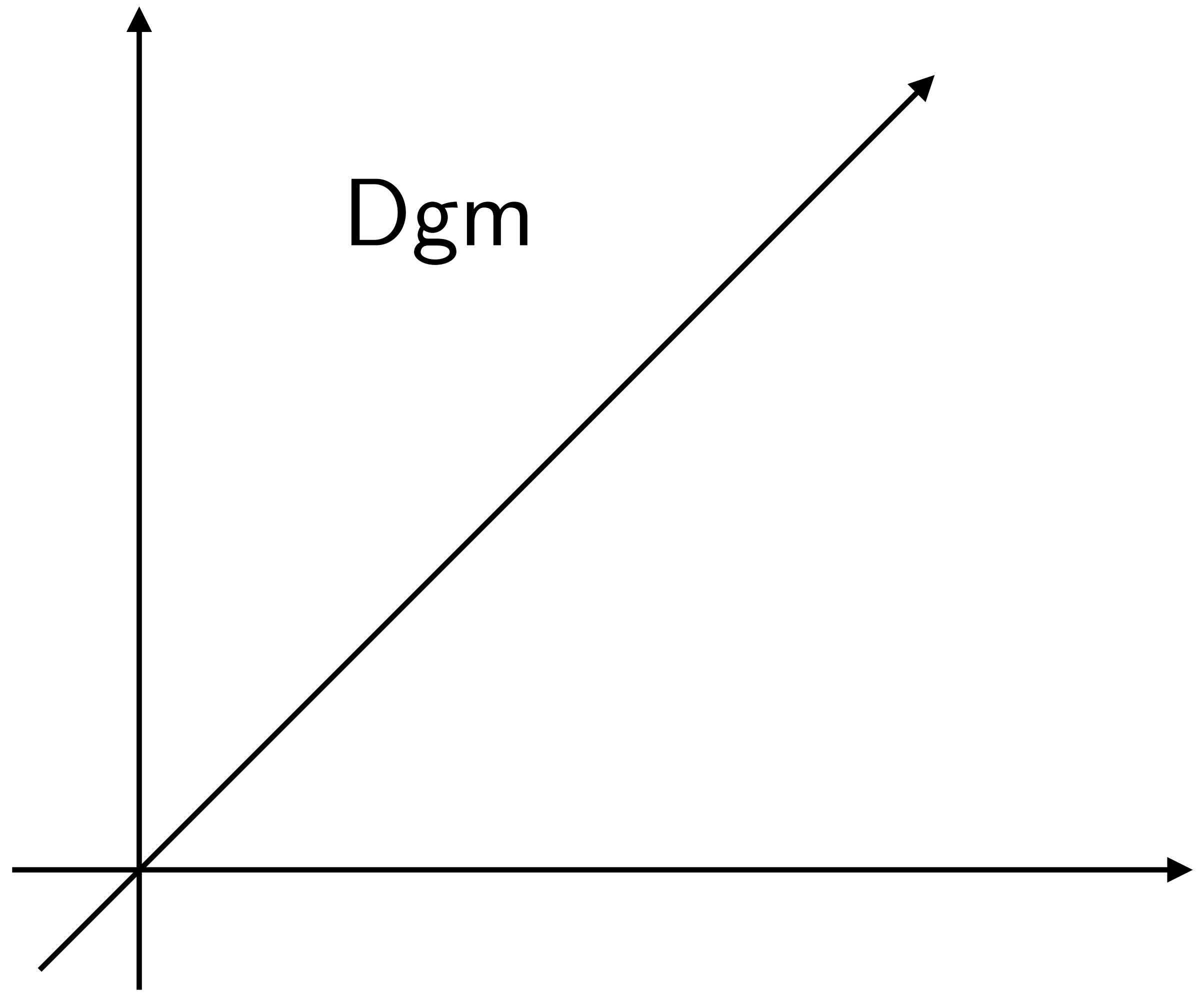
Let $S = \{s_1, \ldots, s_n\}$ be a finite set of real numbers

A persistence module F is S-**constructible** if

- $\mathsf{F}(p) = 0, \text{ for all } p < s_1$
- $\text{for } s_i \leq p \leq q < s_{i+1}, \mathsf{F}(p \leq q) \text{ is an isomorphism}$
- $\text{for } s_n \leq p, \ \mathsf{F}(p \leq q) \text{ is an isomorphism}$



Poset of Intervals

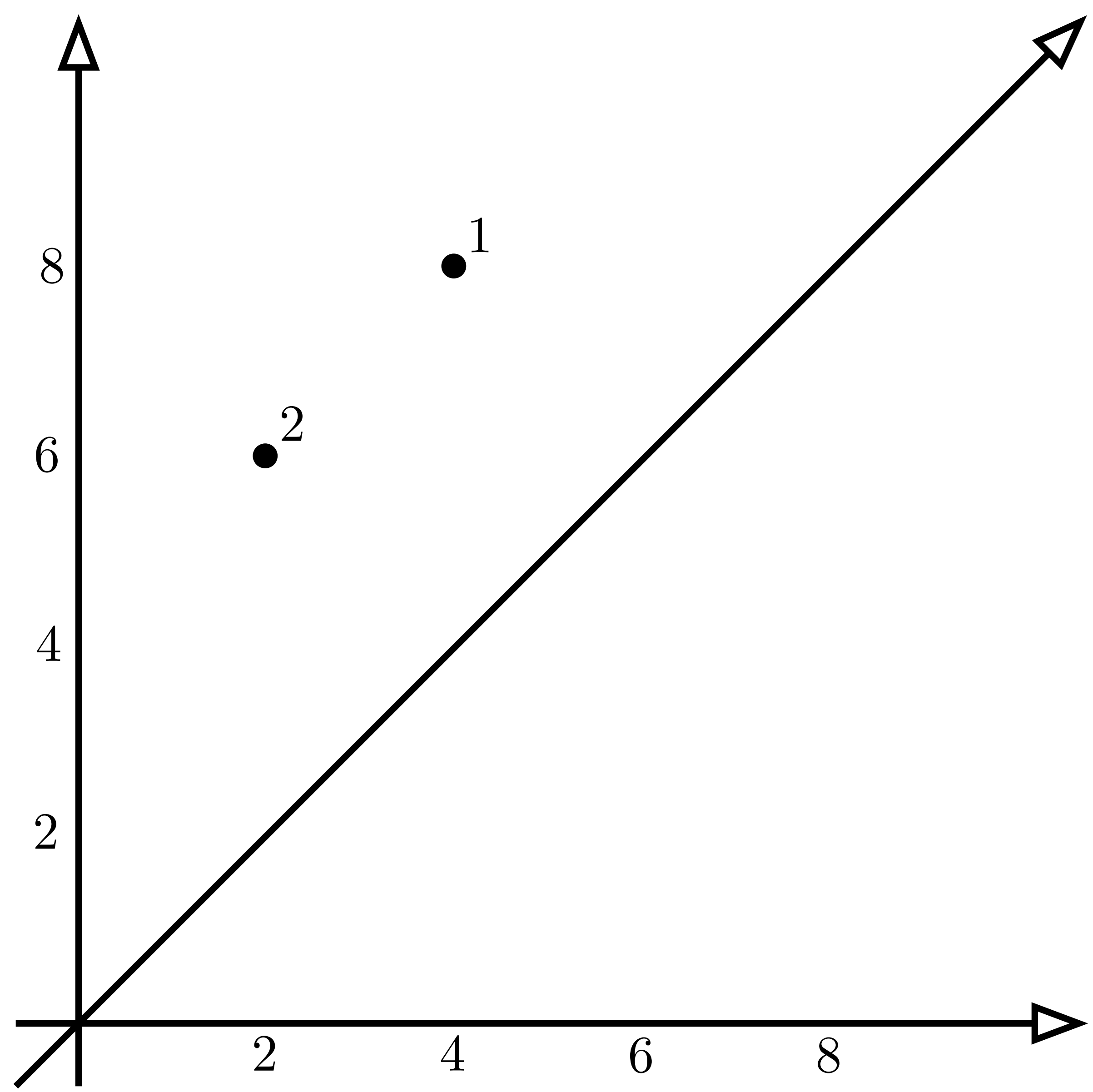


Dgm poset of half-open intervals of the form

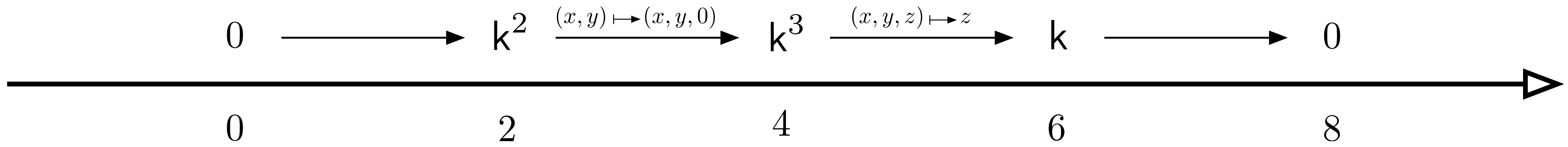
$[r, s)$ and $[r, \infty)$

Persistence Diagram

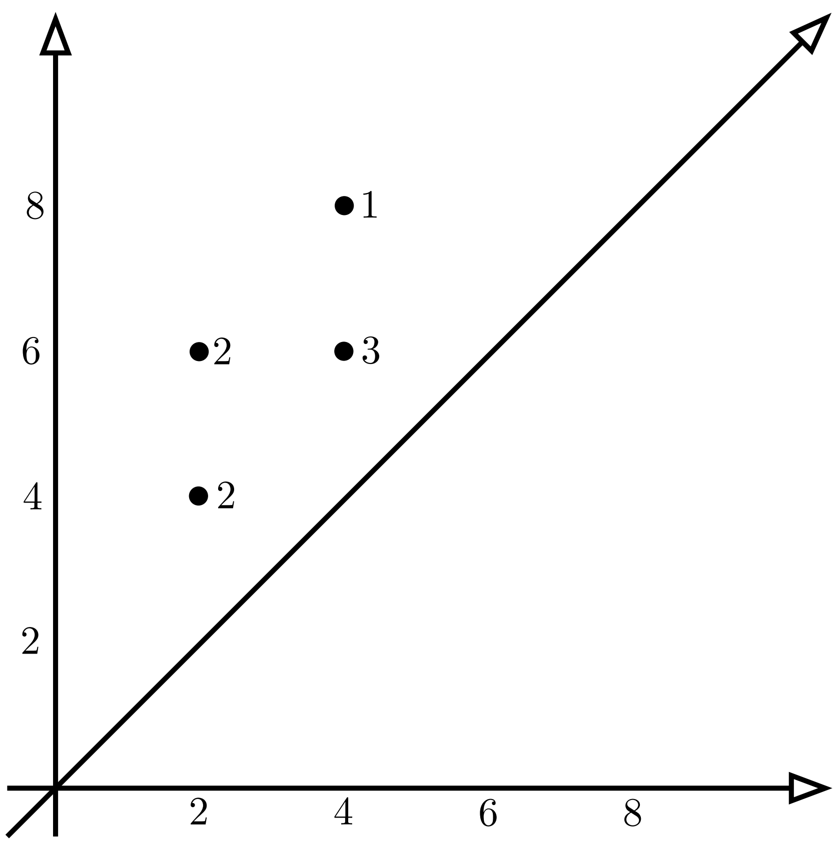
A **persistence diagram** is a map $P : \mathsf{Dgm} \to \mathbb{Z}$ that is nonzero on finitely many intervals

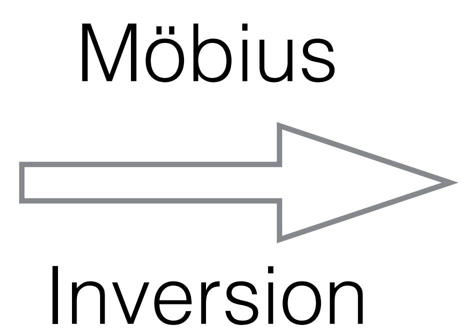


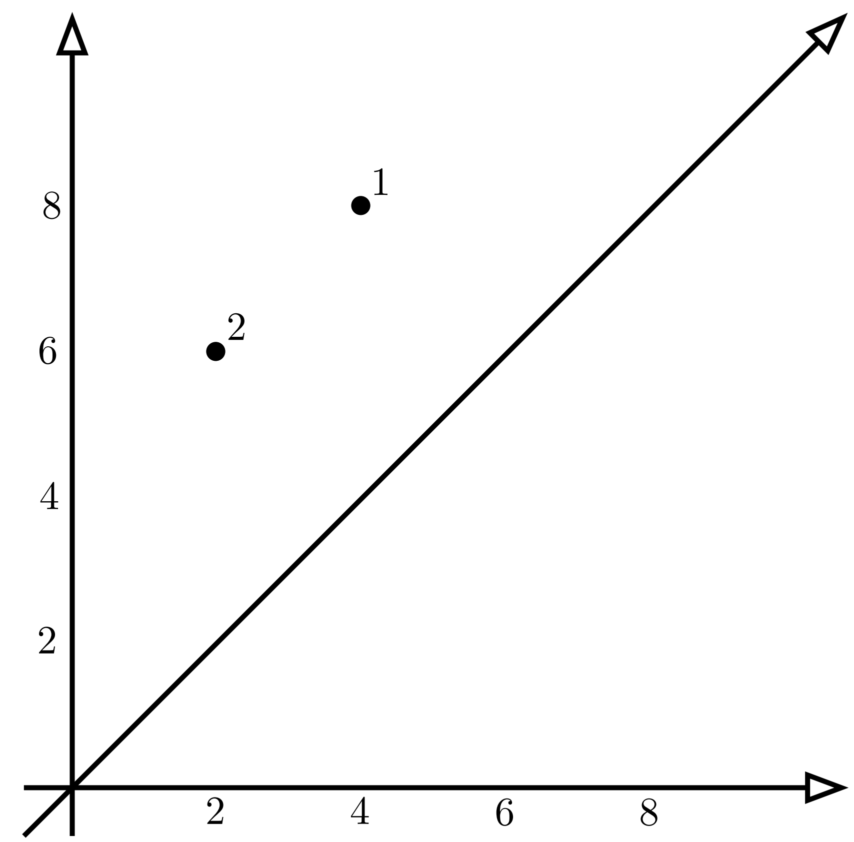
Construction of P. Diagram



$\mathsf{F} : (\mathbb{R}, \leq) \to \mathsf{Vect}$







$d\mathsf{F} : \mathsf{Dgm} \to \mathbb{Z}$

$P_{\mathsf{F}} : \mathsf{Dgm} \to \mathbb{Z}$

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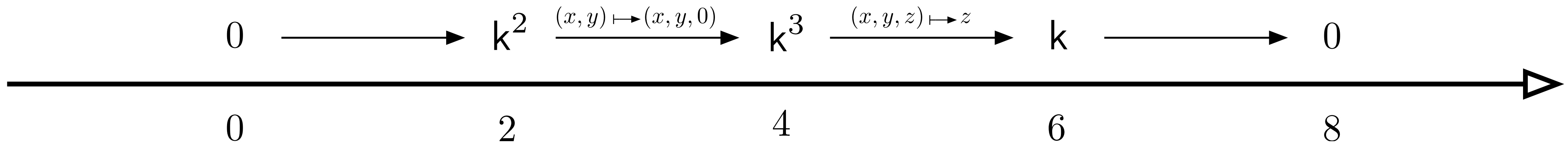

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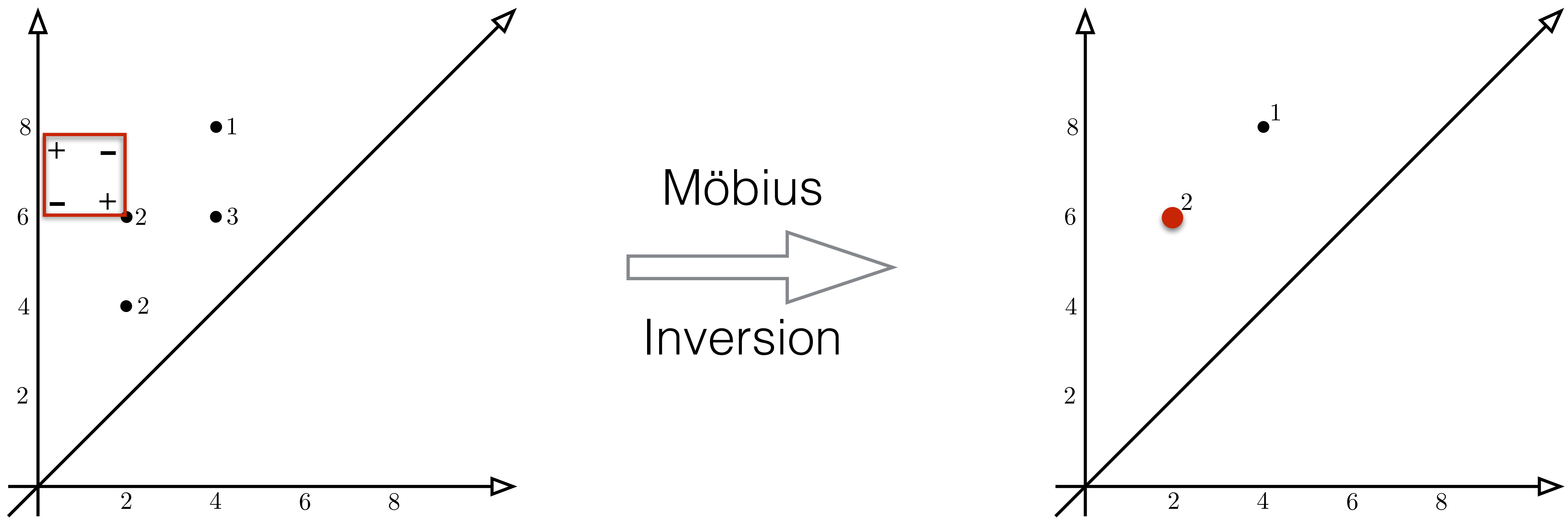

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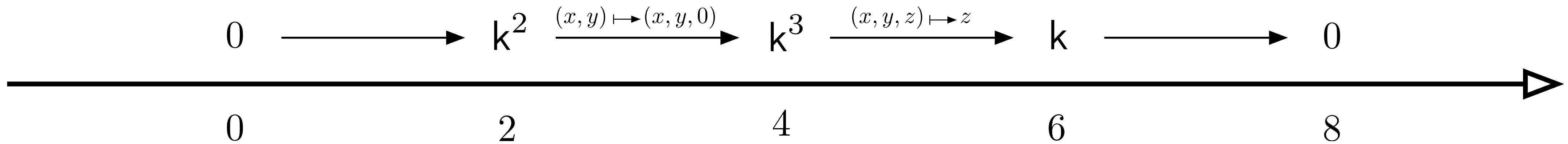

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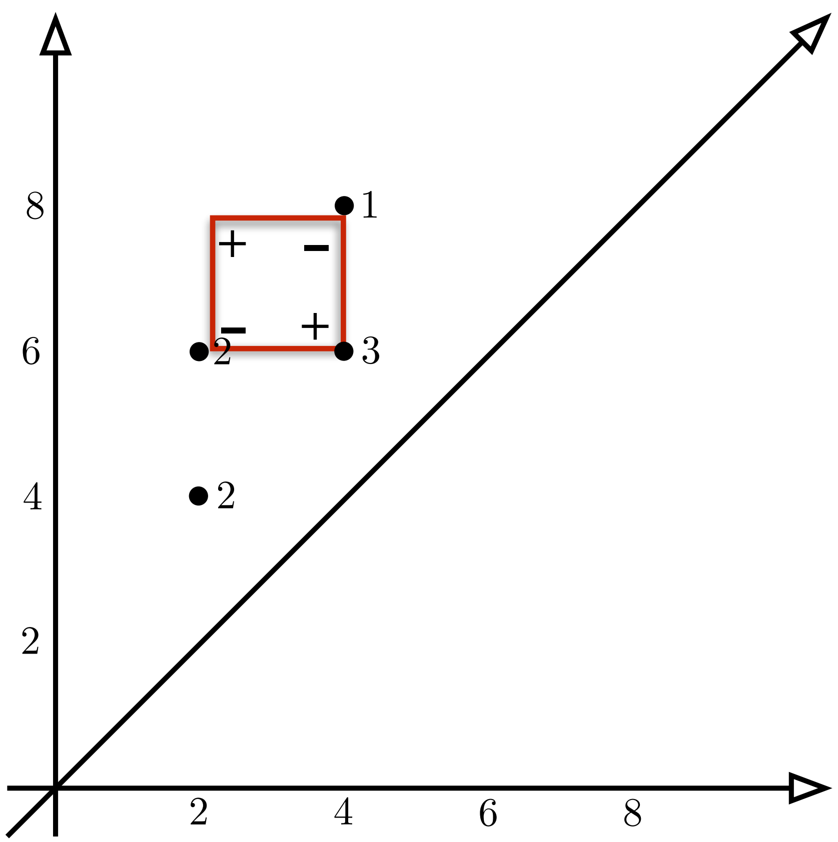
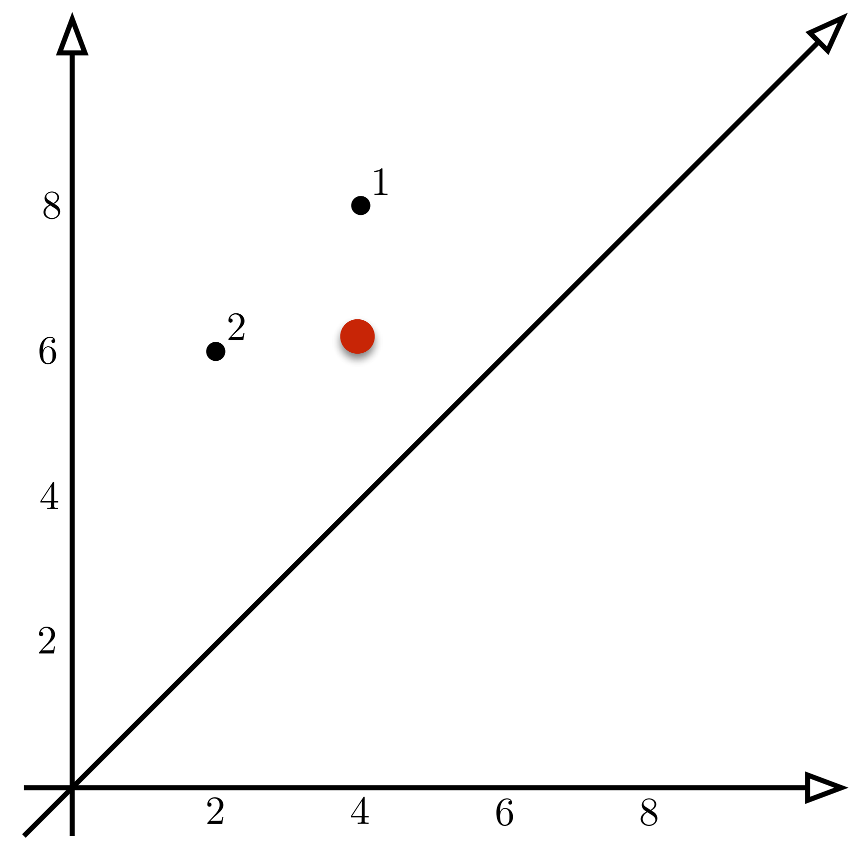


$d\mathsf{F} : \mathsf{Dgm} \to \mathbb{Z}$

Möbius

Inversion



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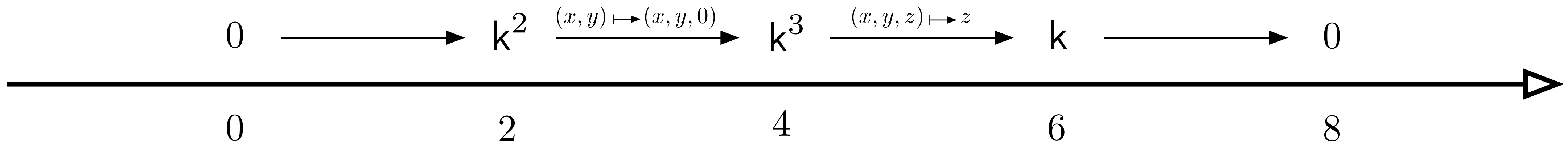

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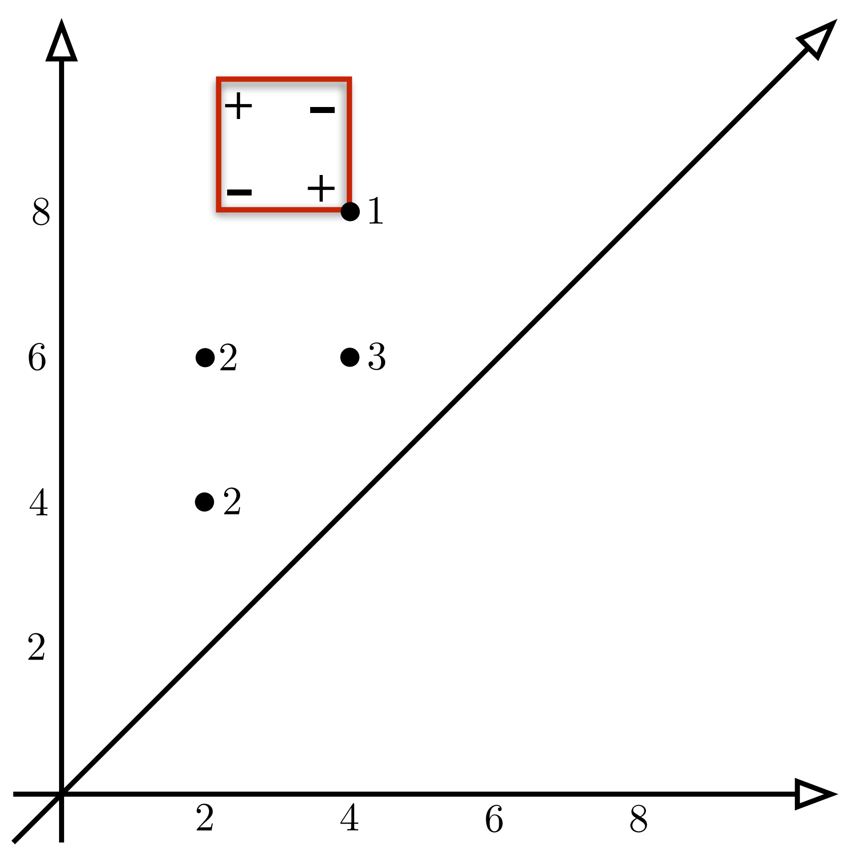

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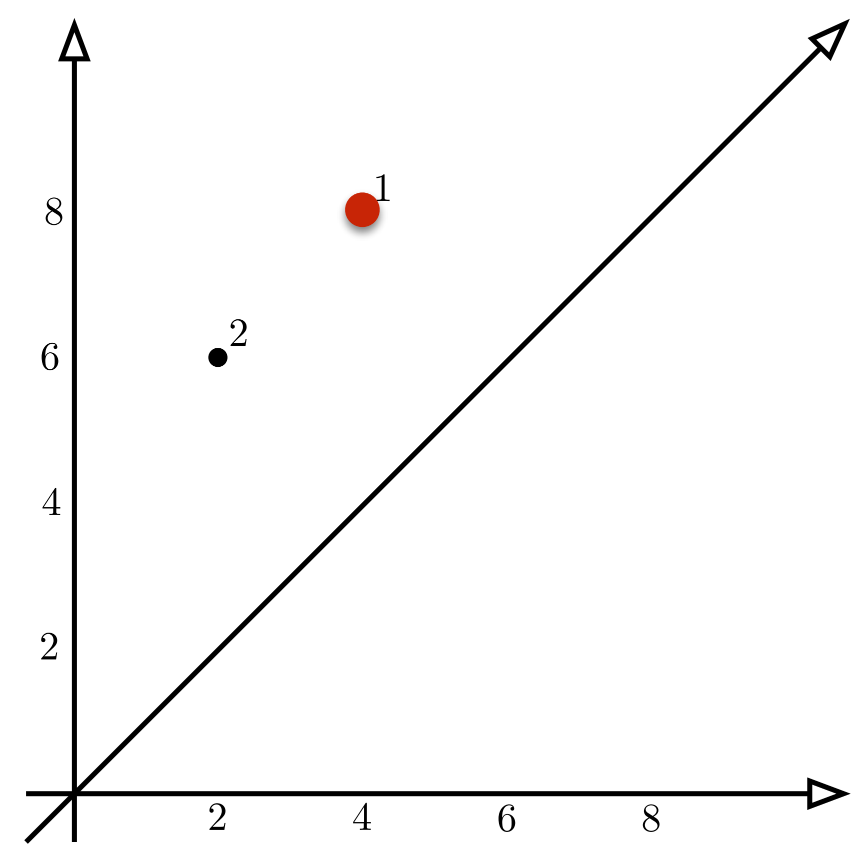

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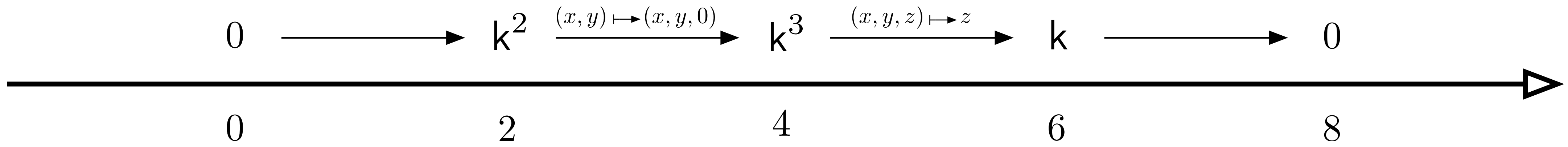

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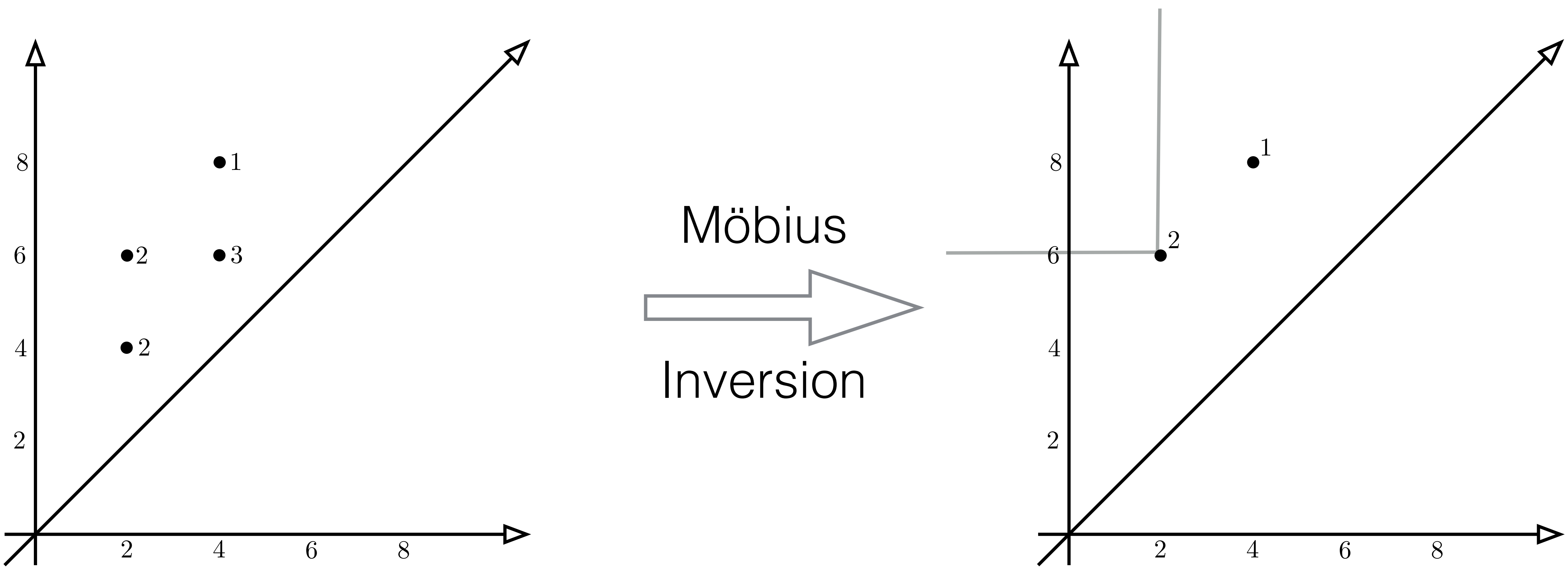

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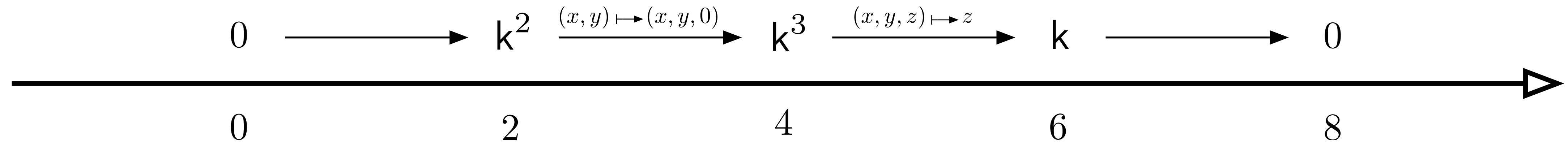

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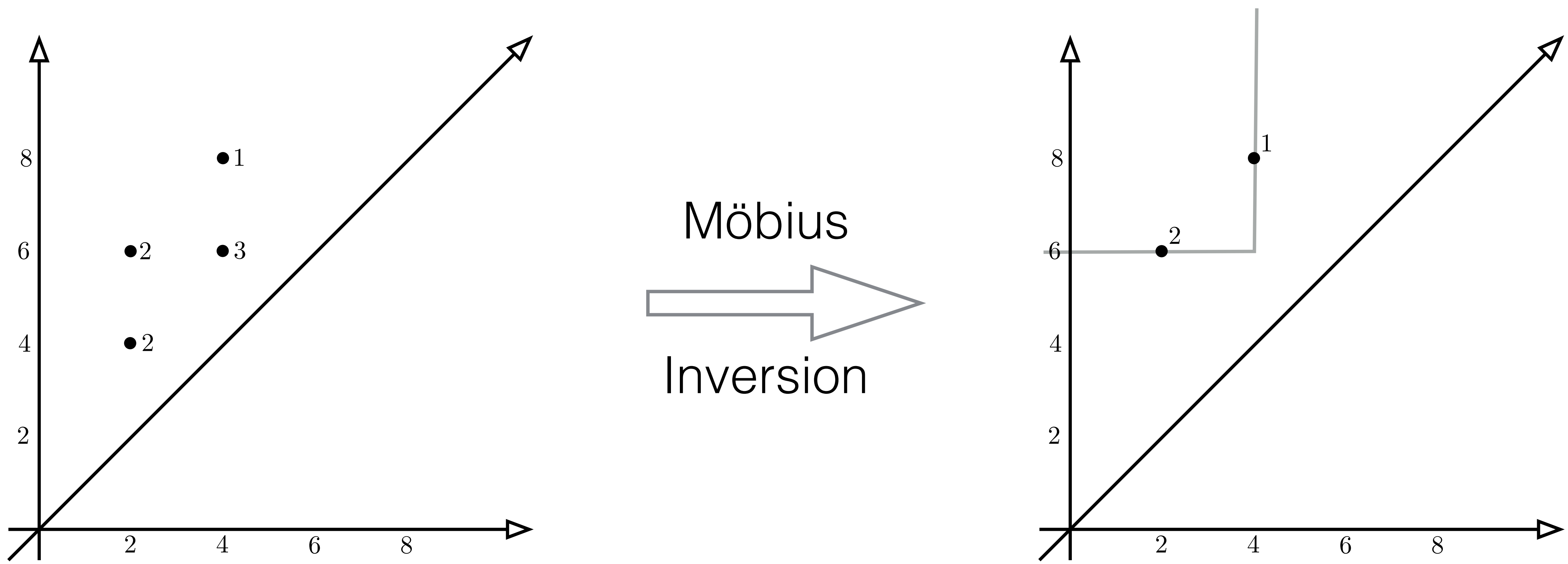

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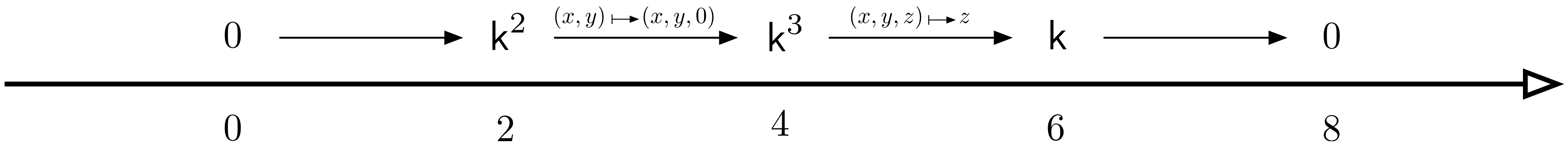

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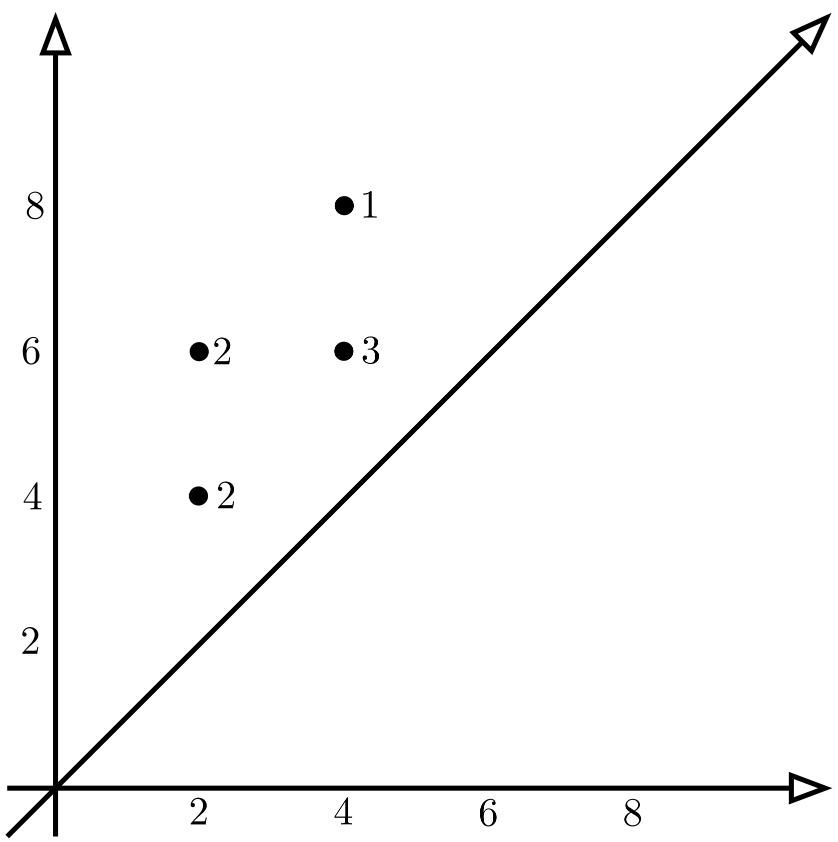

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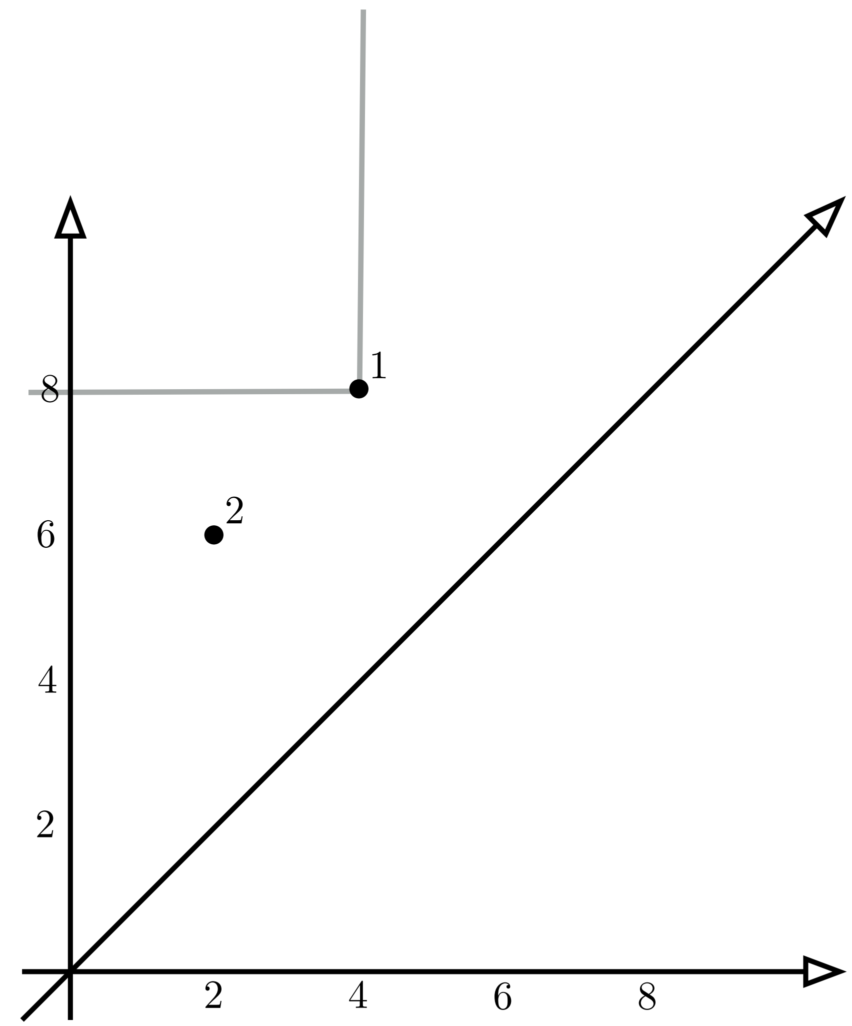

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Persistence and the Möbius Inversion

- The Inclusion-Exclusion construction of the persistence diagram is by

Cohen-Steiner, Edelsbrunner, and Harer. *Stability of persistence* diagrams. 2007.

- Recognized as a Möbius Inversion leading to persistence diagrams for constructible persistence modules valued in small symmetric monoidal categories

A. Patel. *Generalized Persistence Diagrams*. 2018

- There is even bottleneck stability in the case of an abelian category

A. McCleary, A. Patel. Bottleneck Stability for *Generalized Persistence Diagrams*. arxiv

- See also **T. Leinster. *Notions of Möbius Inversion.* 2012**

Maps to Manifolds

$$X \xrightarrow{f} M$$

- We don't have sublevel sets, but we have *level sets* or *fibers* of the map.
- Leray invented sheaf theory to study the fibers of a map

- We believe the persistent homology group is the fundamental unit of persistence.
- For each open set $U \subseteq M$ we ask, what is the fiberwise homology of the map over U that is stable to infinitesimally small perturbations?
- Our answer is the *persistent local system*. It behaves like the persistent homology group.

What is a Local System?

A local system of abelian groups over a manifold is:

A representation of the fundamental groupoid

$$\mathsf{L} : \mathsf{Fund}(M) \to \mathsf{Ab}$$

A locally constant cosheaf

$$\mathsf{L} : \mathsf{Open}(M) \to \mathsf{Ab}$$

A locally constant sheaf

$$\mathsf{L} : \mathsf{Open}^{\mathsf{op}}(M) \to \mathsf{Ab}$$

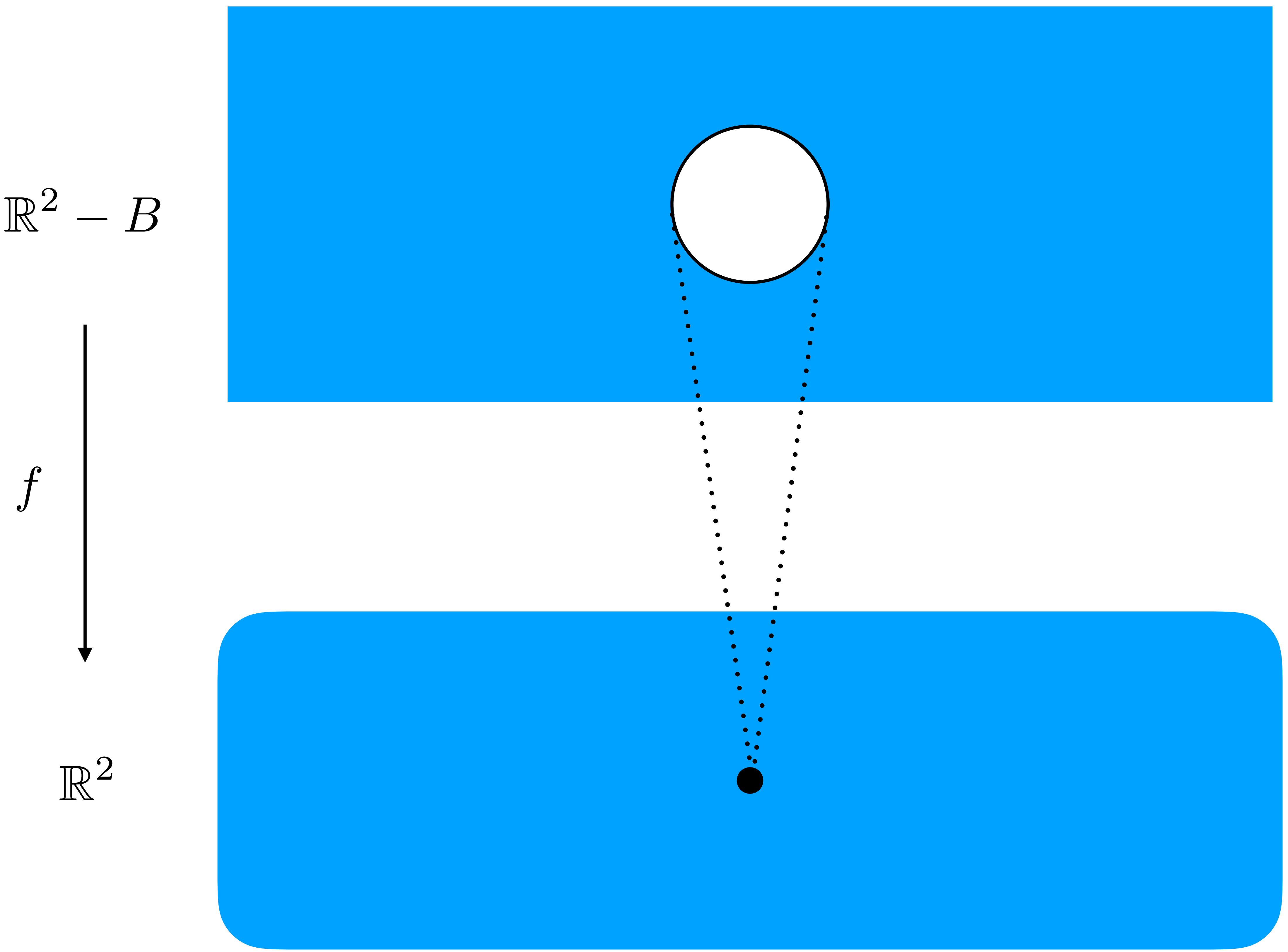
Examples of Local Systems

Take fiberwise one-dimensional homology

L_f is the trivial local system with fiber $\mathbb{Z}$

L_g is the Möbius local system with fiber $\mathbb{Z}$

Fiberwise Stable Homology



Cap Product

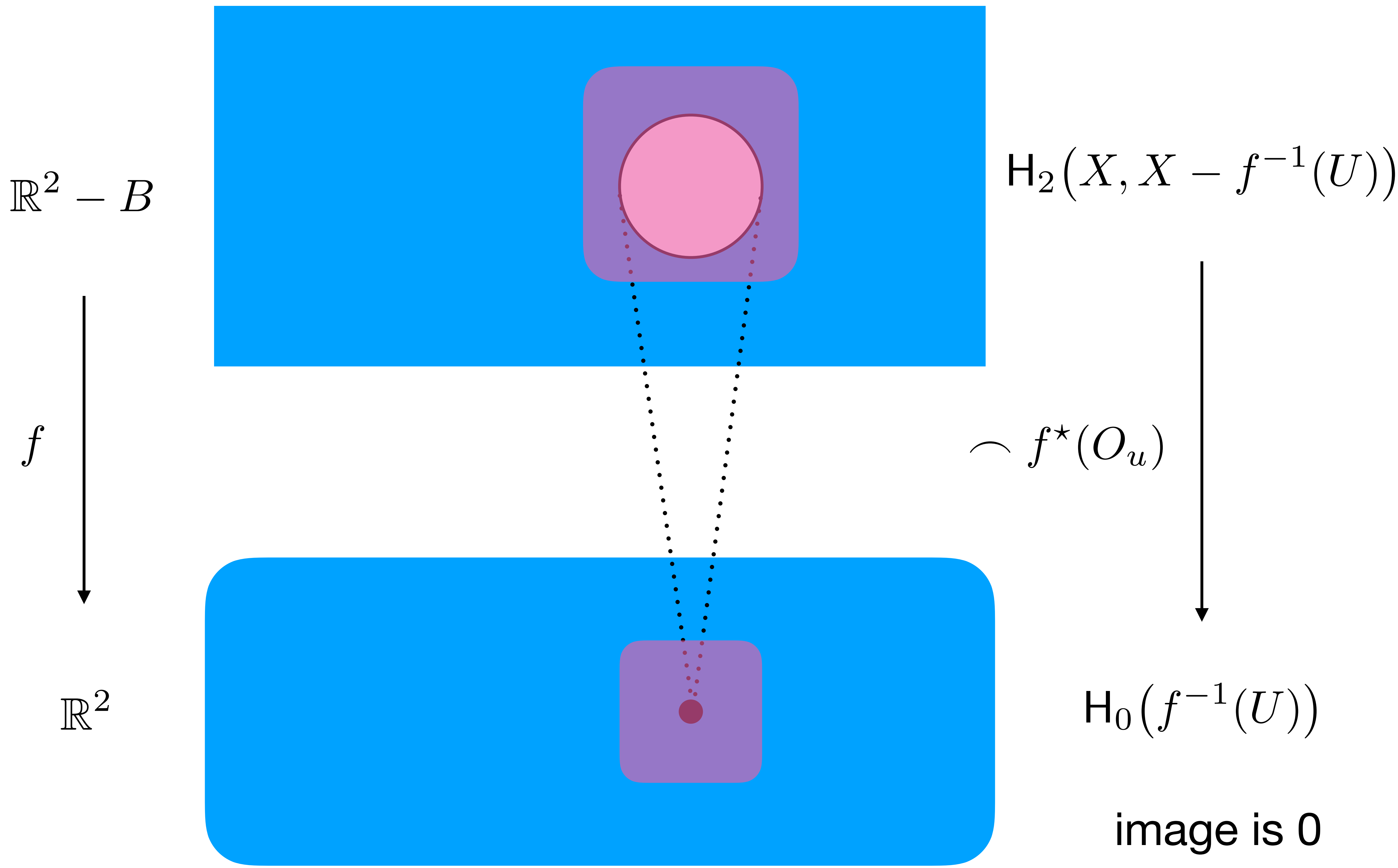
- Given a constructible map $f : X \to M$
- Suppose the target is oriented
- For each point $p \in M$ and a small enough open neighborhood $U \subseteq M$ of p

$$\mathsf{H}_{*+m}\big(X, X - f^{-1}(U)\big) \otimes \mathsf{H}^m\big(X, X - f^{-1}(U)\big) \xrightarrow{\frown} \mathsf{H}_*\big(f^{-1}(U)\big)$$

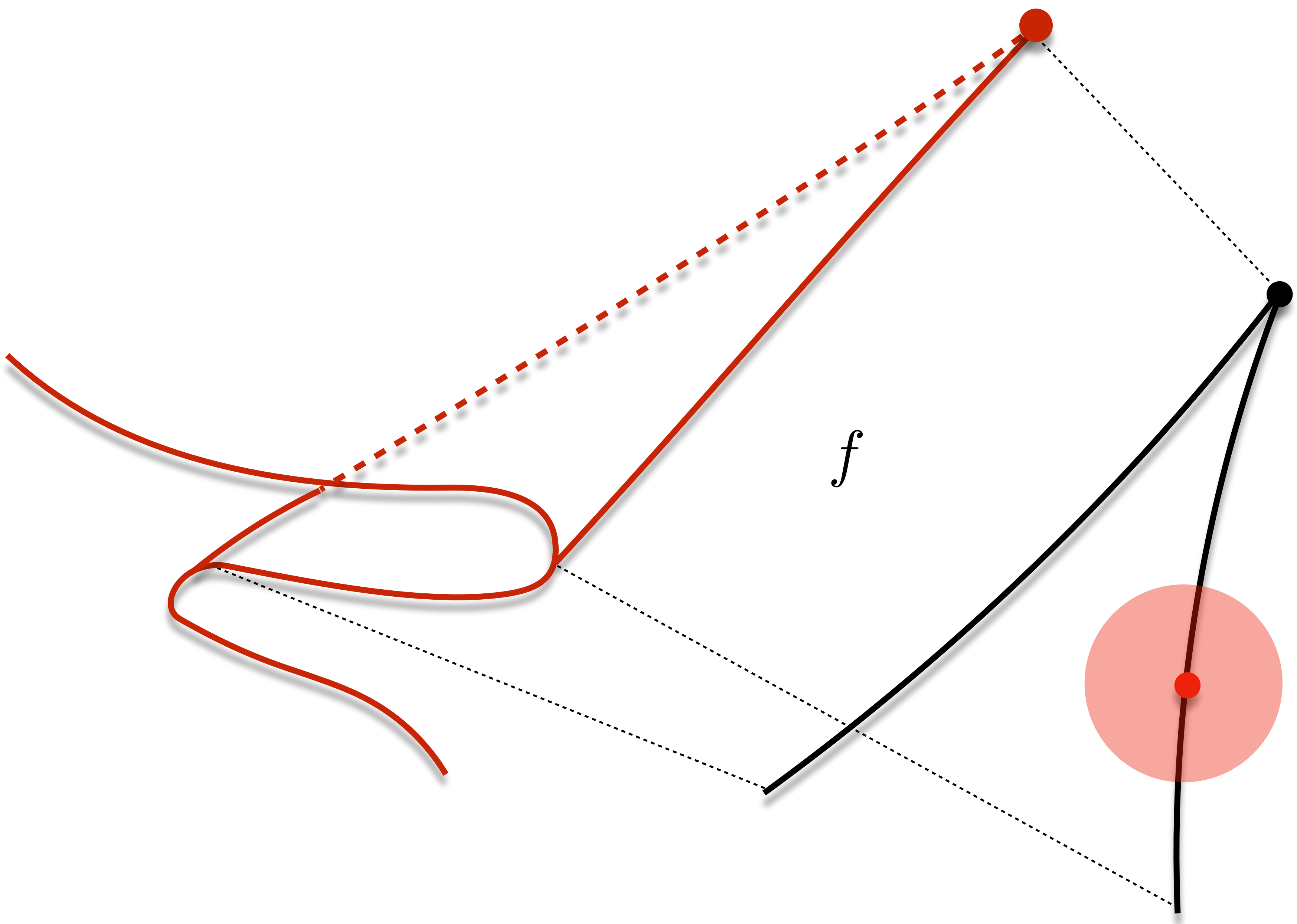
$$\uparrow f^\star$$

$$\mathsf{H}^m(M, M - U) \xrightarrow{\cong} \mathsf{H}^m(M) \cong \mathbb{Z}$$

Fiberwise Stable Homology



Fiberwise Stable Homology



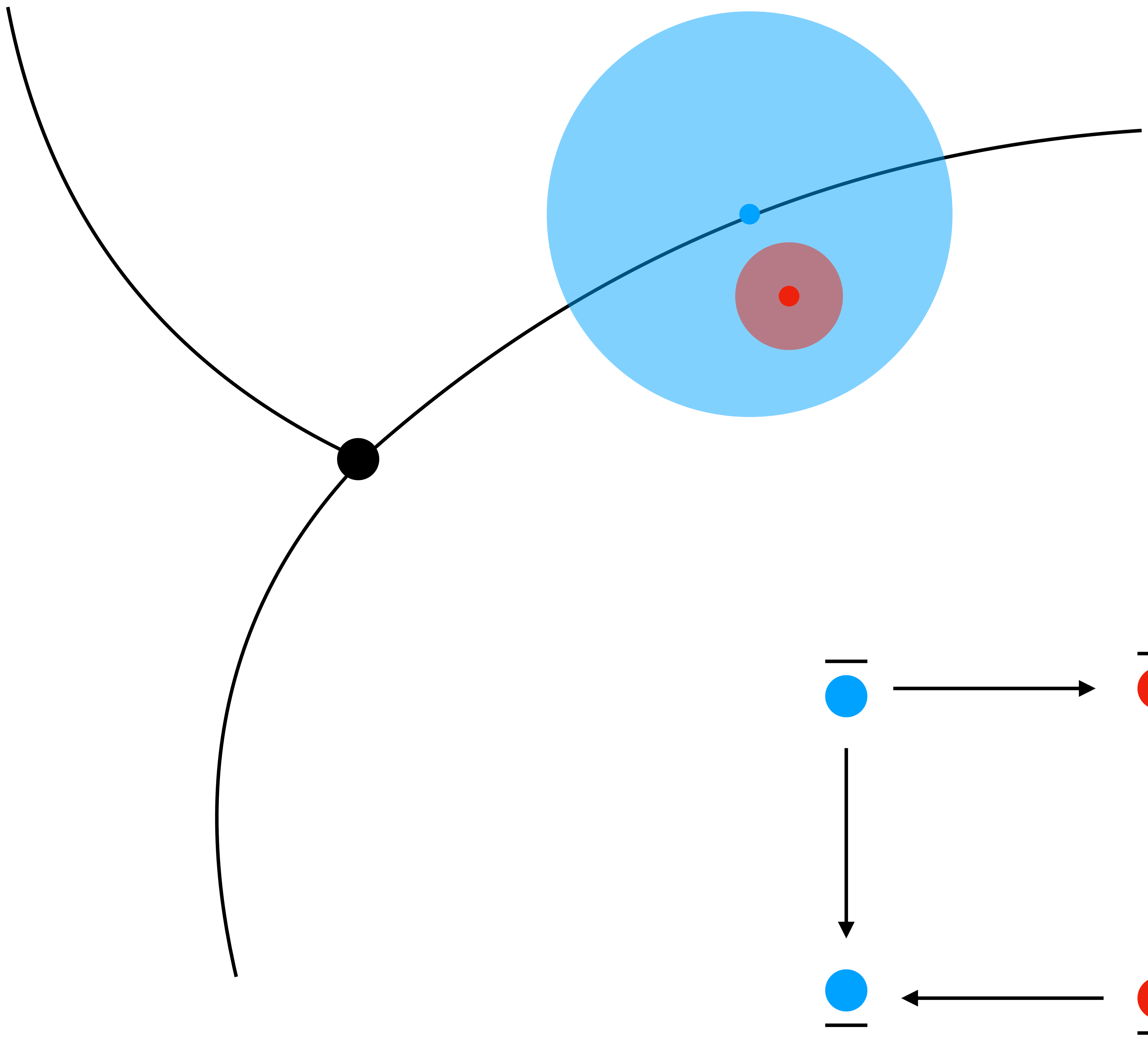
$$\mathsf{H}_2\big(X, X - f^{-1}(U)\big)$$

$$\frown f^{\star}(O_u)$$

$$\mathsf{H}_0\big(f^{-1}(U)\big)$$

Image has rank 1

Bisheaf



Persistence Stack

Given a map $X \xrightarrow{f} M$ to an oriented manifold

we construct some structure $\mathcal{F}_d$ where for each open set $U \subseteq M$

$$\overline{\mathcal{F}}_{d+m}(U) \longleftarrow \cdots\cdots$$ **Episheaf over U (some kind of sheaf over U)**

$$\downarrow$$

$$\underline{\mathcal{F}}_d(U) \longleftarrow \cdots\cdots$$ **Monocosheaf under U (some kind of cosheaf under U)**

The image is the persistent local system over U

Property I

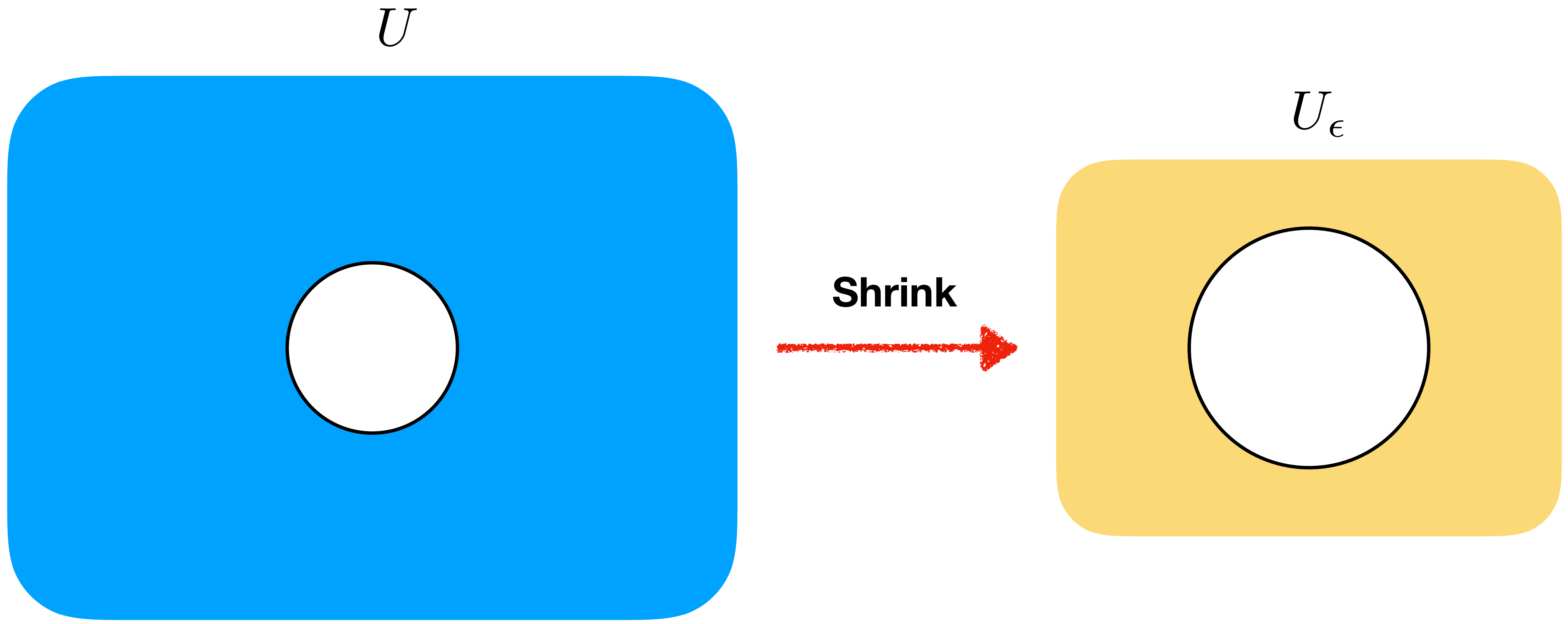
For each inclusion of open sets $i : V \subseteq U$ we have

$$
\begin{array}{ccc}
i^\star \overline{\mathcal{F}}_{d+m}(U) & \longrightarrow & \overline{\mathcal{F}}_{d+m}(V) \\
{\scriptstyle i^\star \mathcal{F}(U)} \downarrow & & \downarrow {\scriptstyle \mathcal{F}(V)} \\
i^\star \underline{\mathcal{F}}_{d}(U) & \longleftarrow & \underline{\mathcal{F}}_{d}(V)
\end{array}
$$

$$\text{rank } i^\star \mathcal{F}(U) \leq \text{rank } \mathcal{F}(V)$$

Property II

For an open set $U \subseteq M$ fix any $\epsilon > 0$ and shrink U by ϵ



Property II

then there is $\delta > 0$ such that for any second constructible map $X \xrightarrow{g} M$

where $\mathsf{Dist}(f, g) < \delta$

$$
\begin{array}{ccc}
i^\star \overline{\mathcal{F}}_{d+m}(U) & \longrightarrow & \overline{\mathcal{G}}_{d+m}(U_\epsilon) \\
i^\star \mathcal{F}(U) \Big\downarrow & {\scriptstyle i : U_\epsilon \subseteq U} & \Big\downarrow \mathcal{G}(U_\epsilon) \\
i^\star \underline{\mathcal{F}}_d(U) & \longleftarrow & \underline{\mathcal{G}}_m(U_\epsilon)
\end{array}
$$

$$\mathrm{rank}\ i^\star \mathcal{F}(U) \leq \mathrm{rank}\ \mathcal{G}(U_\epsilon)$$

Ongoing Work

- Given a constructible map, we create a persistence stack.
- For each open set of the target space, we have a map from an episheaf to a monocosheaf. The image of this map is the **persistent local system** over this open set.
- The persistent local system behaves like the persistent homology group

What is the persistence diagram of a persistence stack?

Thank You

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