Topological Dimensionality Reduction

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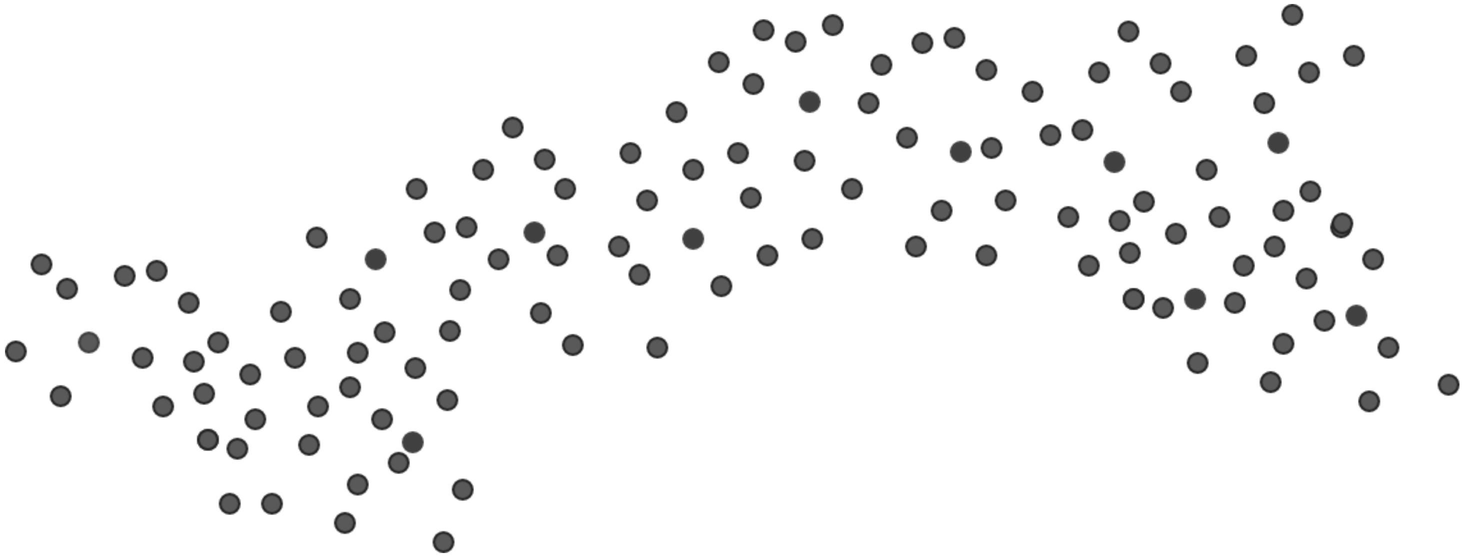
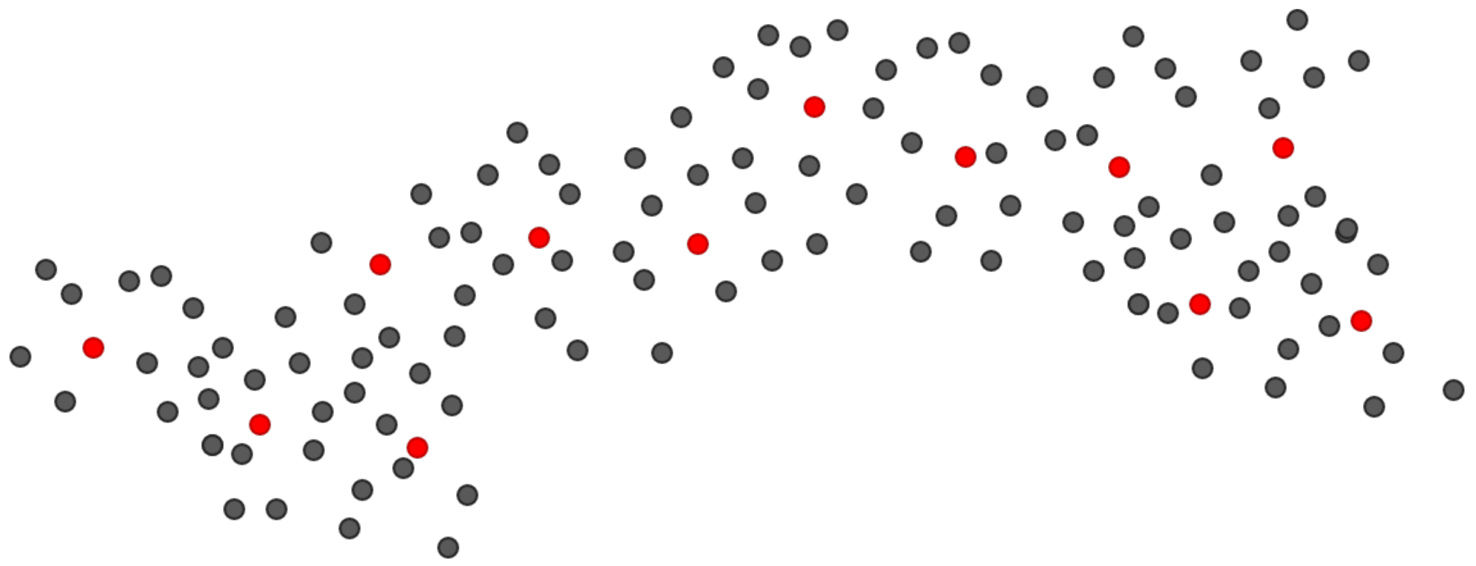
Dpt. of Computational Mathematics, Science & Engineering (CMSE)

Dpt. of Mathematics

$$X \subset (\mathbb{M}, \mathbf{d})$$

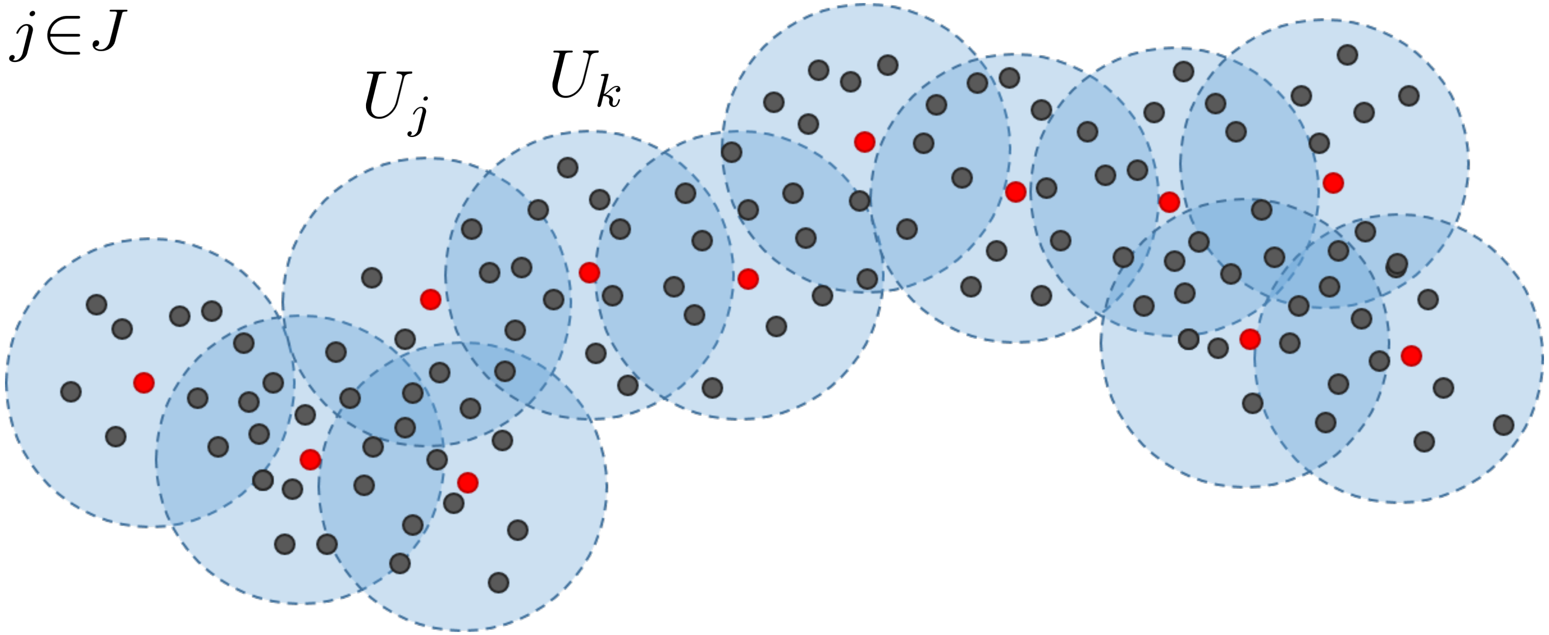


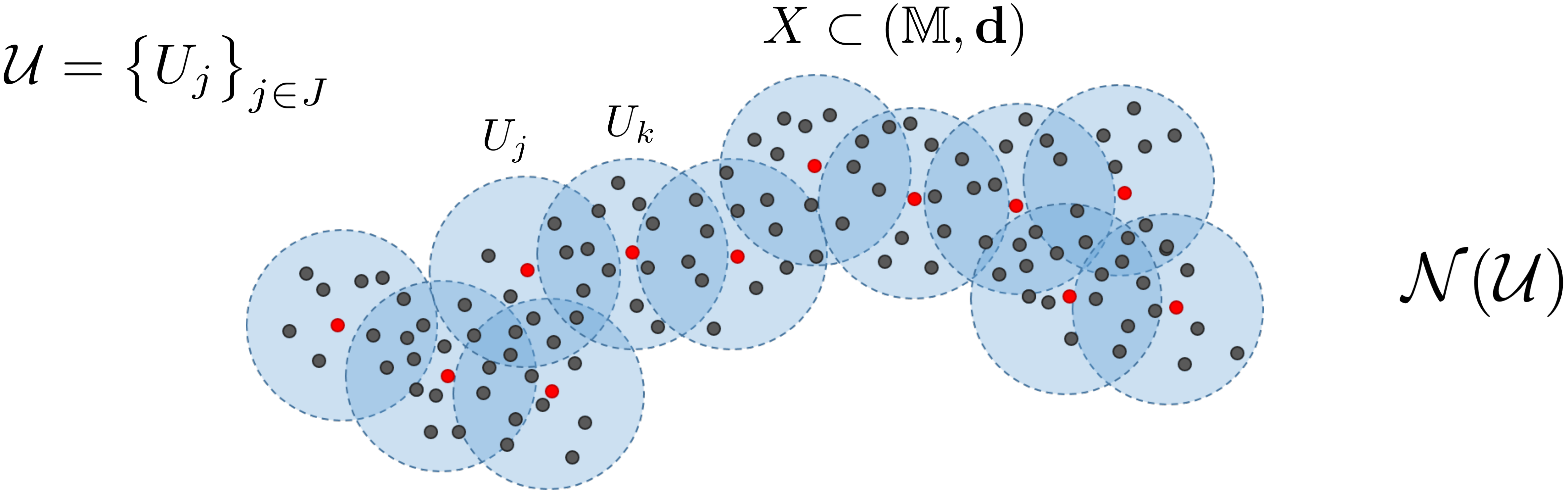
$X \subset (\mathbb{M}, \mathbf{d})$



$$\mathcal{U} = \{U_j\}_{j \in J}$$

$$X \subset (\mathbb{M}, \mathbf{d})$$

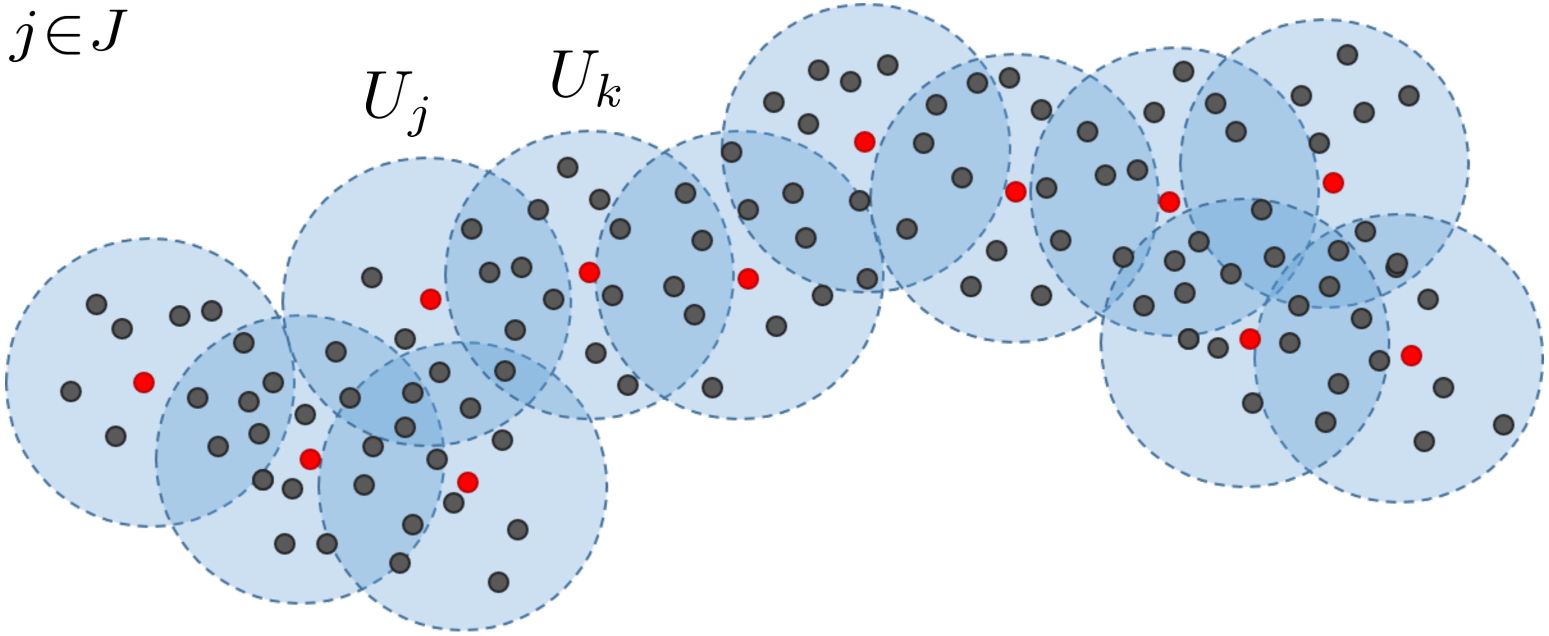



$\mathcal{U} = \{U_j\}_{j \in J}$
$X \subset (\mathbb{M}, \mathbf{d})$
U_j
U_k
$\mathcal{N}(\mathcal{U})$

$$X \subset (\mathbb{M}, \mathbf{d})$$

$$\mathcal{U} = \{U_j\}_{j \in J}$$

$$B = \bigcup \mathcal{U}$$



$$\mathcal{N}(\mathcal{U})$$

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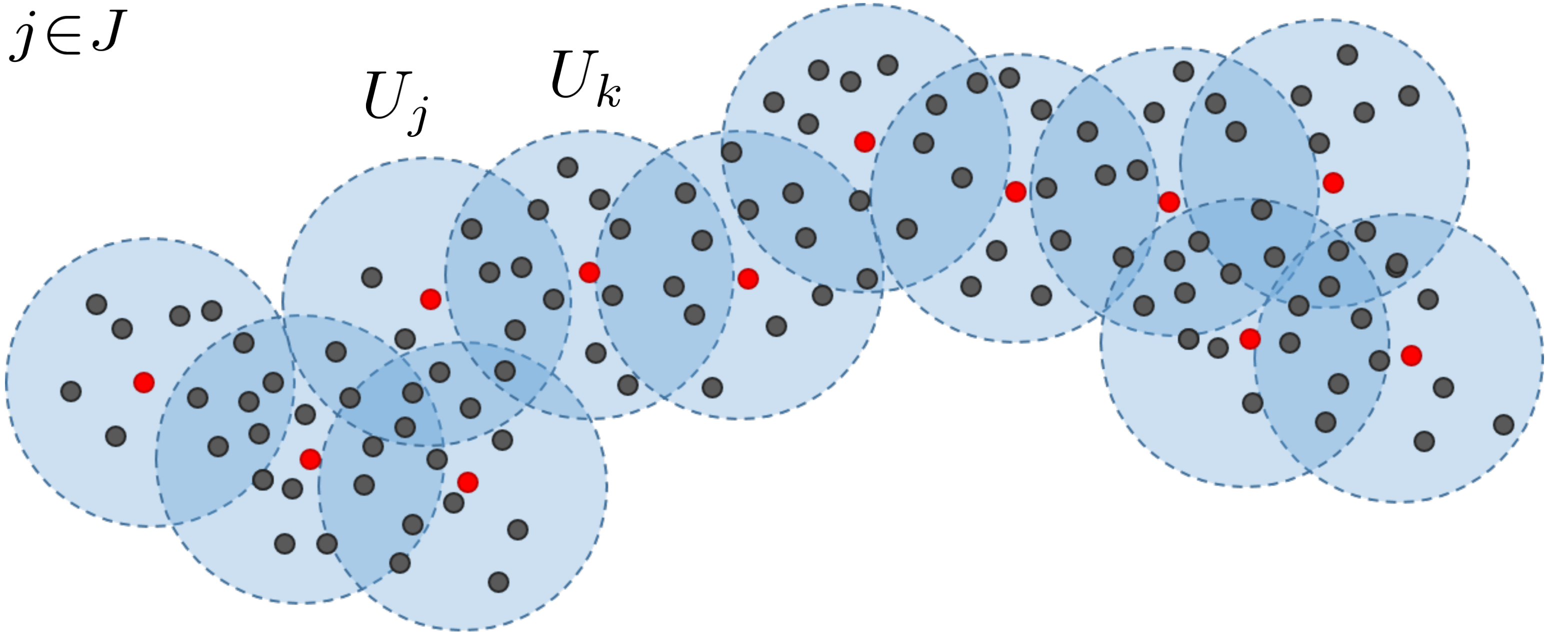

$\mathcal{N}(\mathcal{U})$

Partition of unity:

$\varphi_j : B \longrightarrow [0,1]$

$j \in J$

$\sum_{j \in J} \varphi_j(b) = 1$

$\mathrm{supp}(\varphi_j) \subset \mathrm{clos}(U_j)$

$X \subset (\mathbb{M}, \mathbf{d})$

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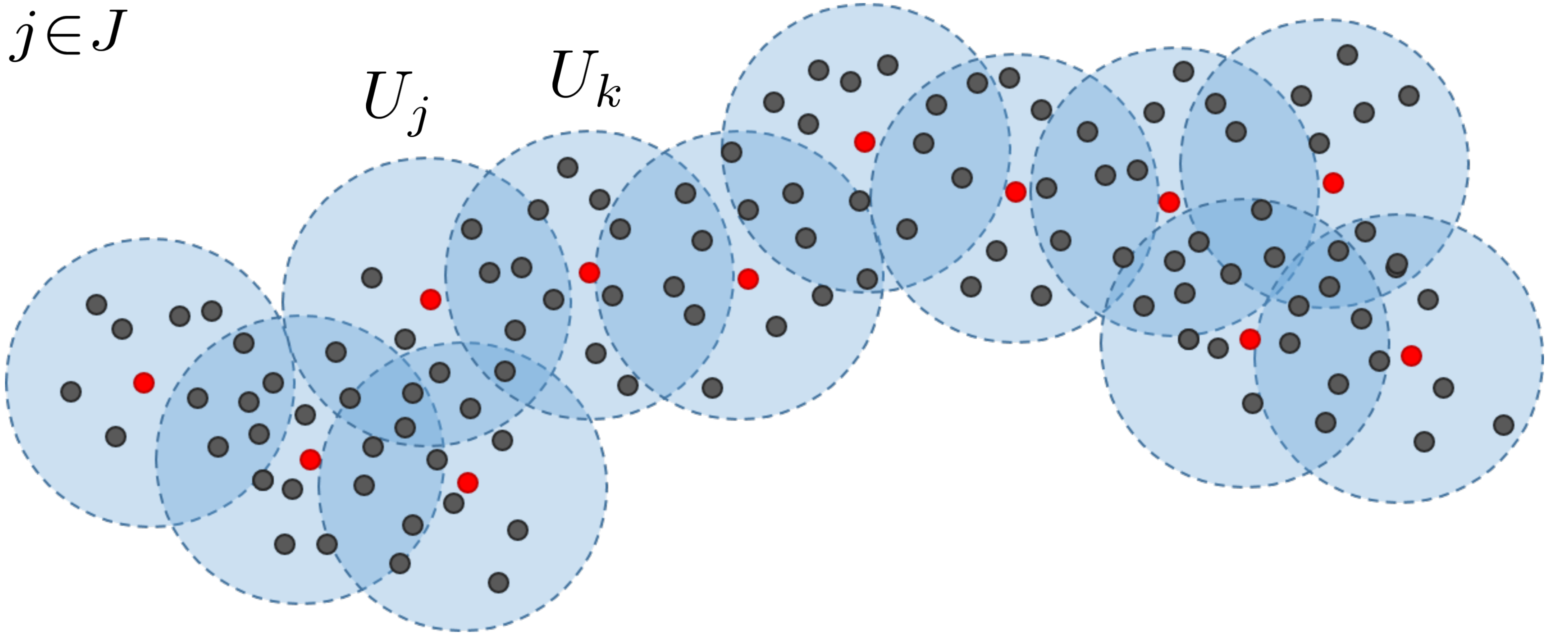
$B = \bigcup \mathcal{U}$



$\{\varphi_j\}_{j \in J}$

Partition of 1

$\mathcal{N}(\mathcal{U})$

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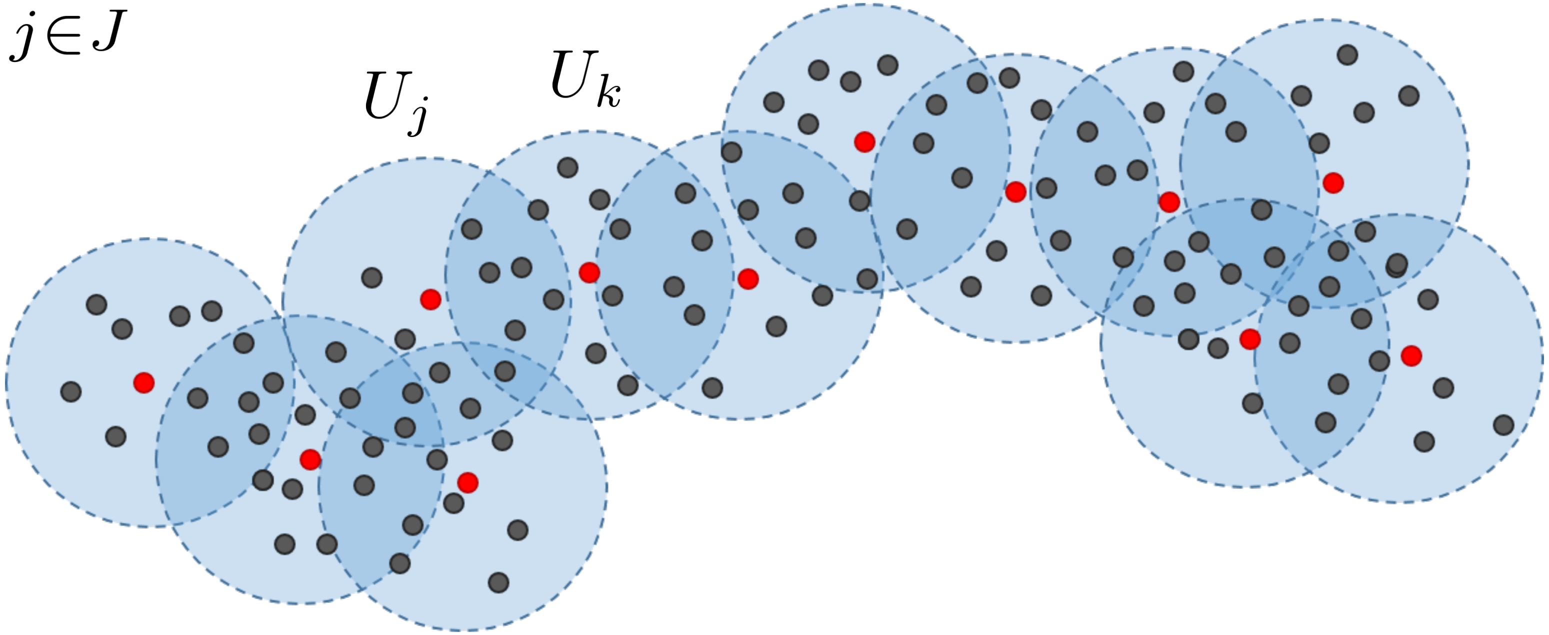


$\{\varphi_j\}_{j \in J}$

Partition of 1

$\mathcal{N}(\mathcal{U})$

<u>Example:</u>

$$
\begin{aligned}
U_j &= \{b \in B : \mathbf{d}(b, x_j) < \alpha\} \\
&= B_\alpha(x_j)
\end{aligned}
$$

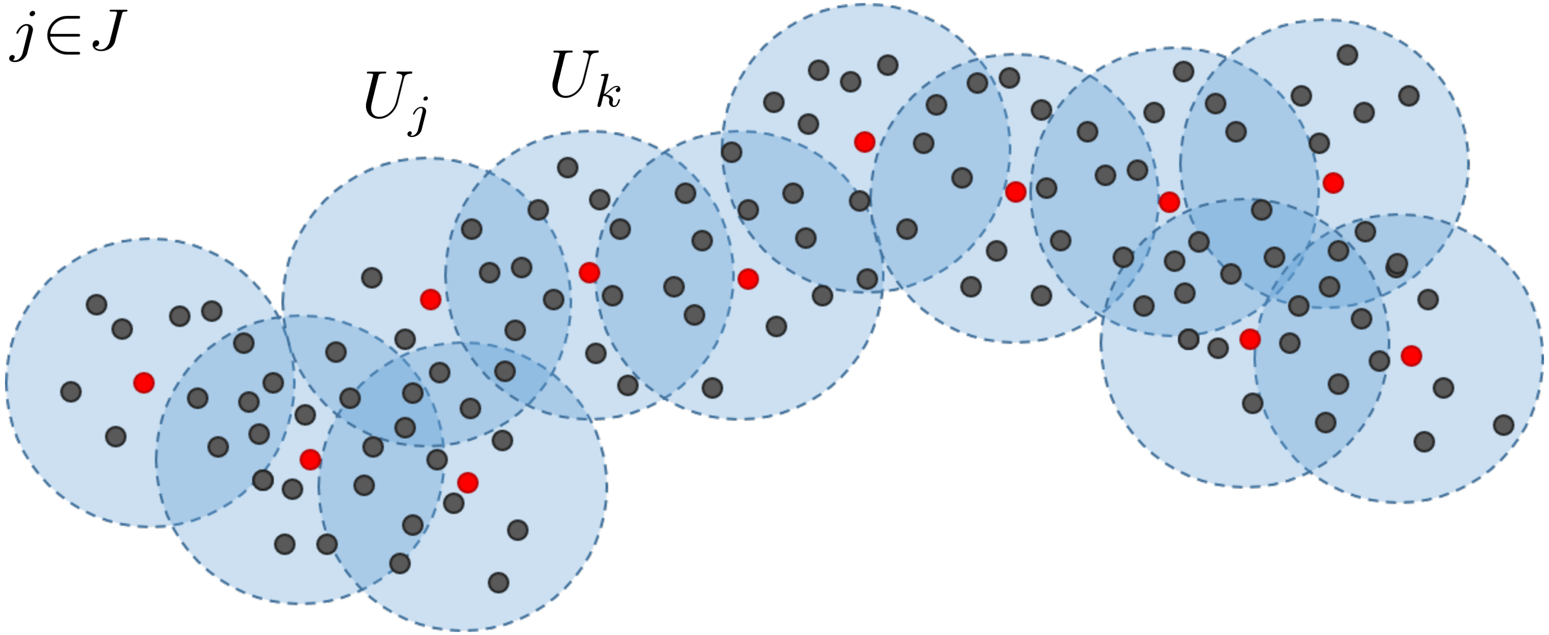
$$X \subset (\mathbb{M}, \mathbf{d})$$

$$\mathcal{U} = \{U_j\}_{j\in J}$$

$$B = \bigcup \mathcal{U}$$

$$\{\varphi_j\}_{j\in J}$$

Partition of 1



$$\mathcal{N}(\mathcal{U})$$

<u>Example:</u>

$$\varphi_j(b) = \frac{|\alpha - \mathbf{d}(b, x_j)|_+}{\sum\limits_{k\in J} |\alpha - \mathbf{d}(b, x_k)|_+}$$

$$U_j = \{b \in B : \mathbf{d}(b, x_j) < \alpha\}$$
$$= B_\alpha(x_j)$$

$$|\lambda|_+ = \max\{\lambda, 0\} \quad , \quad \lambda \in \mathbb{R}$$

Topological Dimensionality Reduction

G

Topological group

G

Topological group

$\mathbb{Z}$

G

Topological group

$\mathbb{Z}$

$S^1 \subset \mathbb{C}$

G Topological group	$\mathbb{Z}$	$S^1 \subset \mathbb{C}$	$\mathbb{Z}/n \subset S^1$

G Topological group	$\mathbb{Z}$	$S^1 \subset \mathbb{C}$	$\mathbb{Z}/n \subset S^1$	$O_k(\mathbb{F})$

G Topological group	$\mathbb{Z}$	$S^1 \subset \mathbb{C}$	$\mathbb{Z}/n \subset S^1$	$O_k(\mathbb{F})$
	$P \longrightarrow B$ Fiber bundle			

G Topological group	$\mathbb{Z}$	$S^1 \subset \mathbb{C}$	$\mathbb{Z}/n \subset S^1$	$O_k(\mathbb{F})$

$$P \longrightarrow B$$

Fiber bundle

+

$$P \times G \longrightarrow P$$

fiberwise free and transitive

G Topological group	$\mathbb{Z}$	$S^1 \subset \mathbb{C}$	$\mathbb{Z}/n \subset S^1$	$O_k(\mathbb{F})$
$\mathrm{Prin}_G(B)$ Isomorphism classes of Principal *G*-bundles	$P \longrightarrow B$ Fiber bundle	$+$	$P \times G \longrightarrow P$ fiberwise free and transitive	

Theorem: Let $\mathscr{C}_G$ be the sheaf (over B) of continuous G-valued functions; $U \subset B$, $\mathscr{C}_G(U) = \text{Maps}(U, G)$, then

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$$\text{Prin}_G(B) \cong \check{H}^1(B; \mathscr{C}_G)$$

G Topological group	$\mathbb{Z}$	$S^1 \subset \mathbb{C}$	$\mathbb{Z}/n \subset S^1$	$O_k(\mathbb{F})$
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Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

$\text{Prin}_G(B)$

Isomorphism classes of Principal *G*-bundles

$P \longrightarrow B$

Fiber bundle

+

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Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

G discrete $\Rightarrow$ $BG \simeq K(G,1)$

$\mathrm{Prin}_G(B)$ Isomorphism classes of Principal *G*-bundles	$P \longrightarrow B$ Fiber bundle + $P \times G \longrightarrow P$ fiberwise free and transitive

Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

G discrete $\Rightarrow$ $BG \simeq K(G,1)$

$G \simeq K(H,n)$ $\Rightarrow$ $BG \simeq K(H,n+1)$

$\text{Prin}_G(B)$	$P \longrightarrow B$	+	$P \times G \longrightarrow P$
Isomorphism classes of Principal *G*-bundles	Fiber bundle		fiberwise free and transitive

Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

G	$\mathbb{Z}$	$\mathbb{Z}/2$	S^1	$\mathbb{Z}/n$	$O_n(\mathbb{F})$
EG					
BG					

$\text{Prin}_G(B)$	$P \longrightarrow B$ + $P \times G \longrightarrow P$
Isomorphism classes of Principal *G*-bundles	Fiber bundle + fiberwise free and transitive

Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

G	$\mathbb{Z}$	$\mathbb{Z}/2$	S^1	$\mathbb{Z}/n$	$O_n(\mathbb{F})$
EG					
BG	S^1	$\mathbb{RP}^\infty$		L_n^∞	

$\text{Prin}_G(B)$	$P \longrightarrow B$	+	$P \times G \longrightarrow P$
Isomorphism classes of Principal *G*-bundles	Fiber bundle		fiberwise free and transitive

Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

G	$\mathbb{Z}$	$\mathbb{Z}/2$	S^1	$\mathbb{Z}/n$	$O_n(\mathbb{F})$
EG	$\mathbb{R}$	$S^\infty \subset \mathbb{R}^\infty$		$S^\infty \subset \mathbb{C}^\infty$	
BG	S^1	$\mathbb{R}\mathbf{P}^\infty$		L_n^∞	

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$\text{Prin}_G(B)$	$P \longrightarrow B$	+	$P \times G \longrightarrow P$
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<u>Universal Bundle:</u> $EG \longrightarrow BG$ $\quad EG \sim \star$

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EG	$\mathbb{R}$	$S^\infty \subset \mathbb{R}^\infty$	$S^\infty \subset \mathbb{C}^\infty$	$S^\infty \subset \mathbb{C}^\infty$	$V_n(\mathbb{F}^\infty)$
BG	S^1	$\mathbb{R}\mathbf{P}^\infty$	$\mathbb{C}\mathbf{P}^\infty$	L_n^∞	$G_n(\mathbb{F}^\infty)$

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Universal Bundle: $EG \longrightarrow BG$ $EG \sim \star$

Theorem: $[B, BG] \cong \text{Prin}_G(B)$

$\text{Prin}_G(B)$

Isomorphism classes of Principal *G*-bundles

$P \longrightarrow B$

Fiber bundle

+

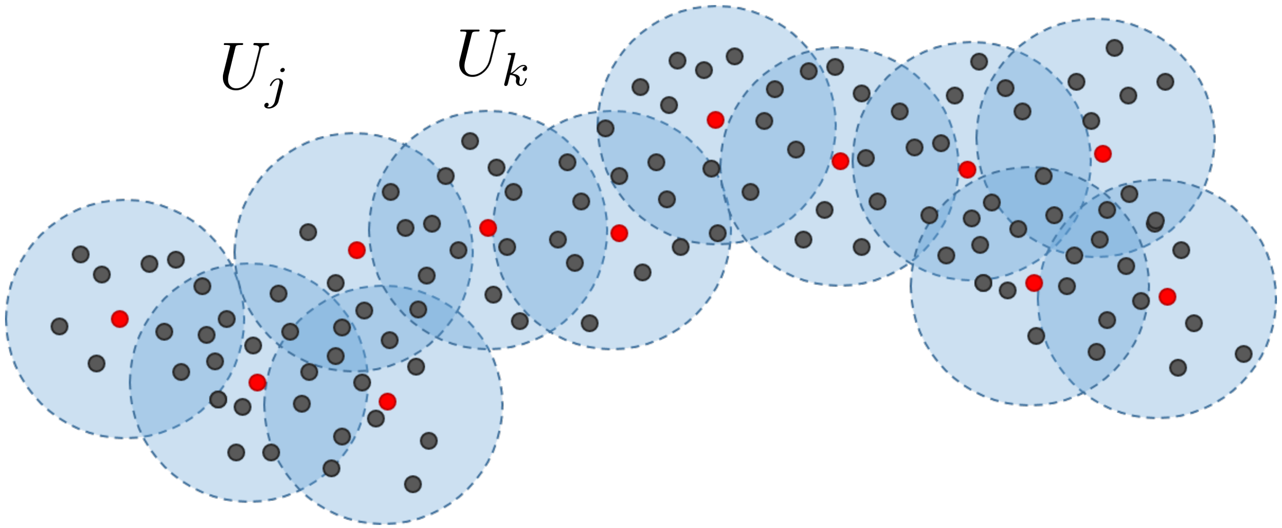
$P \times G \longrightarrow P$

fiberwise free and transitive

$$\mathcal{U} = \{U_j\}_{j \in J}$$

Partition of 1

$$\{\varphi_j\}_{j \in J}$$



$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

$\mathsf{Prin}_G(B)$

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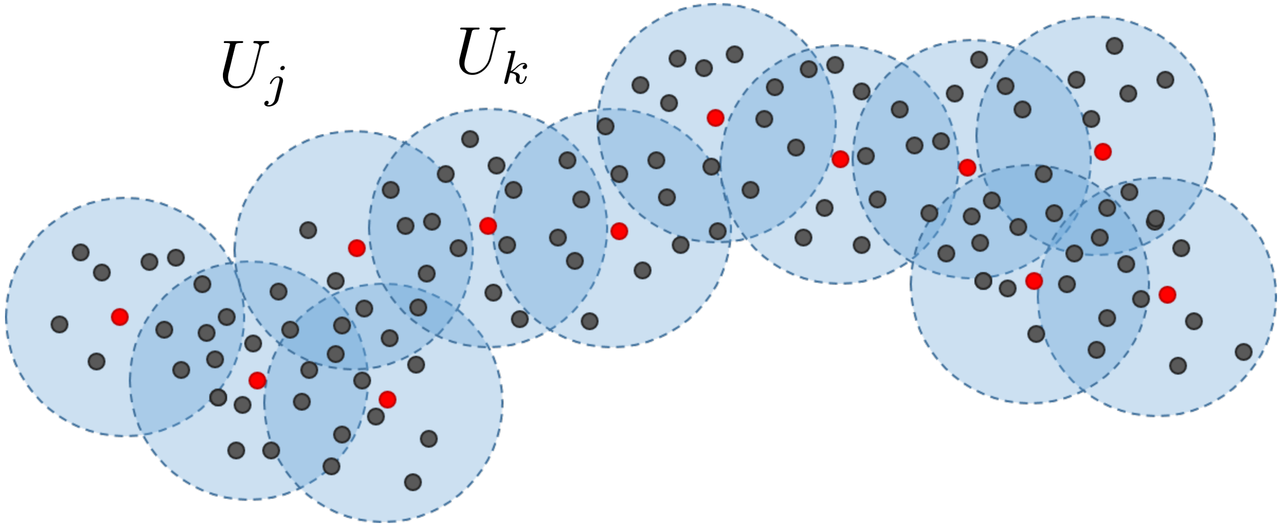

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$$\mathrm{Prin}_G(B) \xleftarrow{\cong} [B, BG]$$

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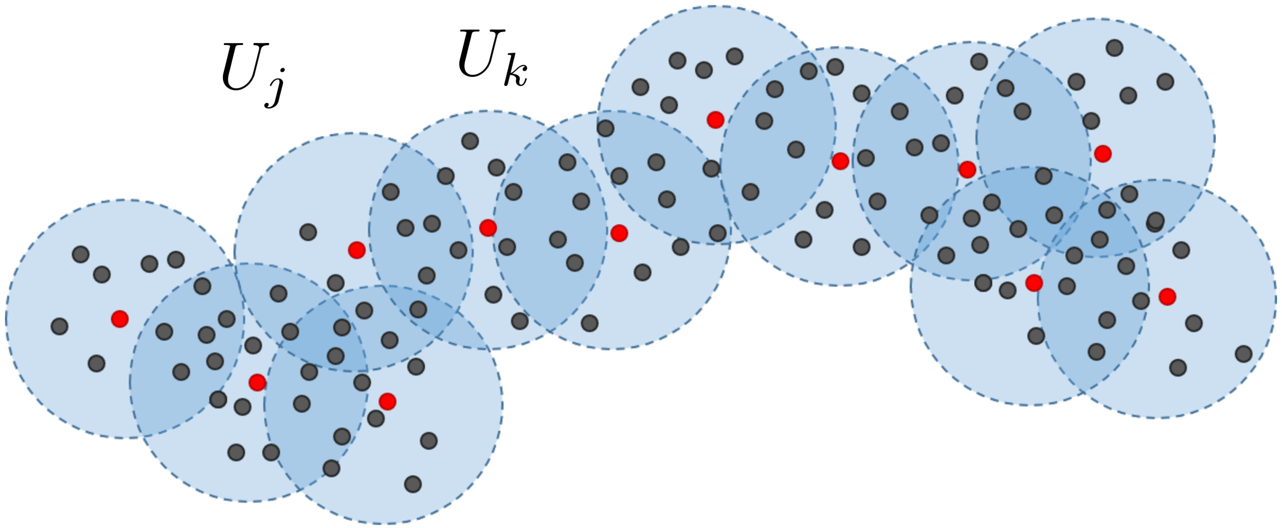
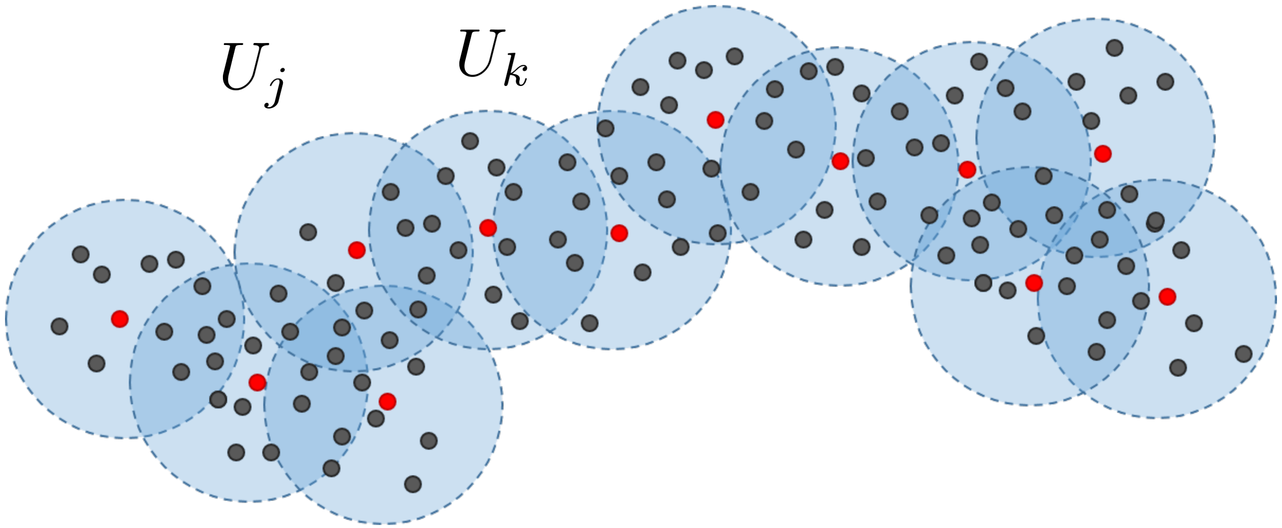


$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

$$\check{H}^1(B;\mathscr{C}_G) \xleftarrow{\cong} \mathsf{Prin}_G(B) \xleftarrow{\cong} [B, BG]$$

$$\mathcal{U} = \{U_j\}_{j\in J}$$



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$$\check{H}^1(\mathcal{U};\mathscr{C}_G) \longrightarrow \check{H}^1(B;\mathscr{C}_G) \xleftarrow{\cong} \mathsf{Prin}_G(B) \xleftarrow{\cong} [B, BG]$$

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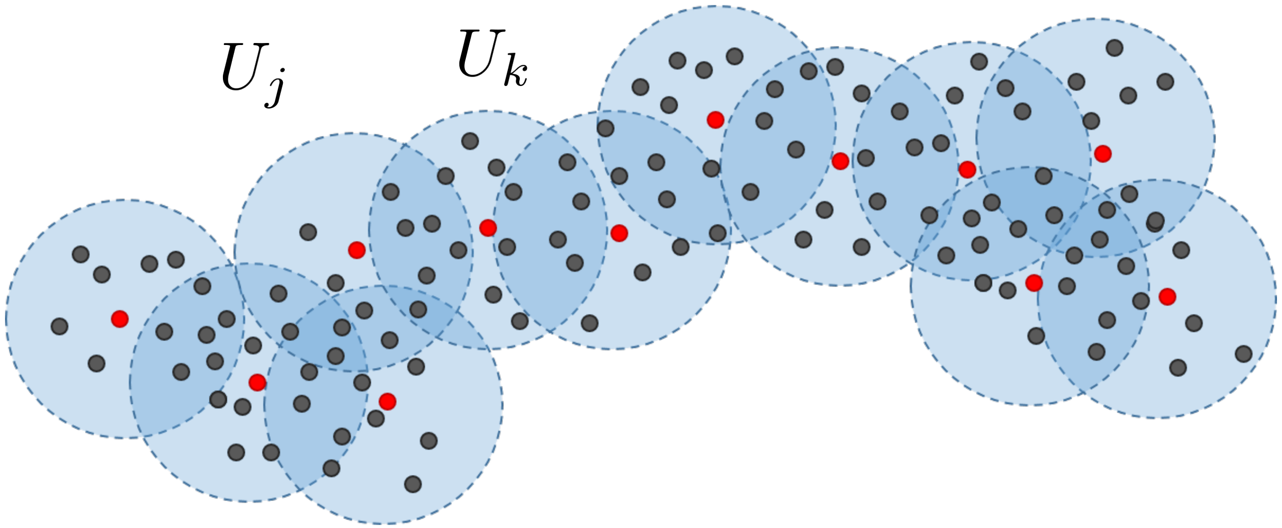

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$$[\{f_{jk}\}]$$

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Partition of 1

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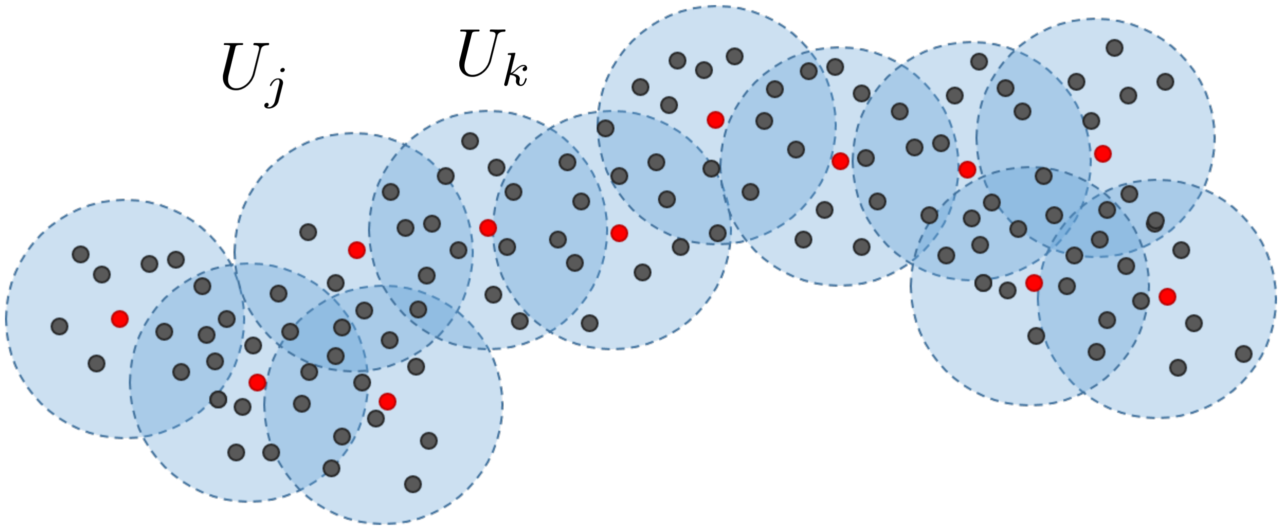

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$$\check{H}^1(\mathcal{U};\mathscr{C}_G) \longrightarrow \check{H}^1(B;\mathscr{C}_G) \overset{\cong}{\longleftarrow} \mathsf{Prin}_G(B) \overset{\cong}{\longleftarrow} [B, BG]$$

$[\{f_{jk}\}]$ $\qquad$ $[f]$

$\mathcal{U} = \{U_j\}_{j\in J}$

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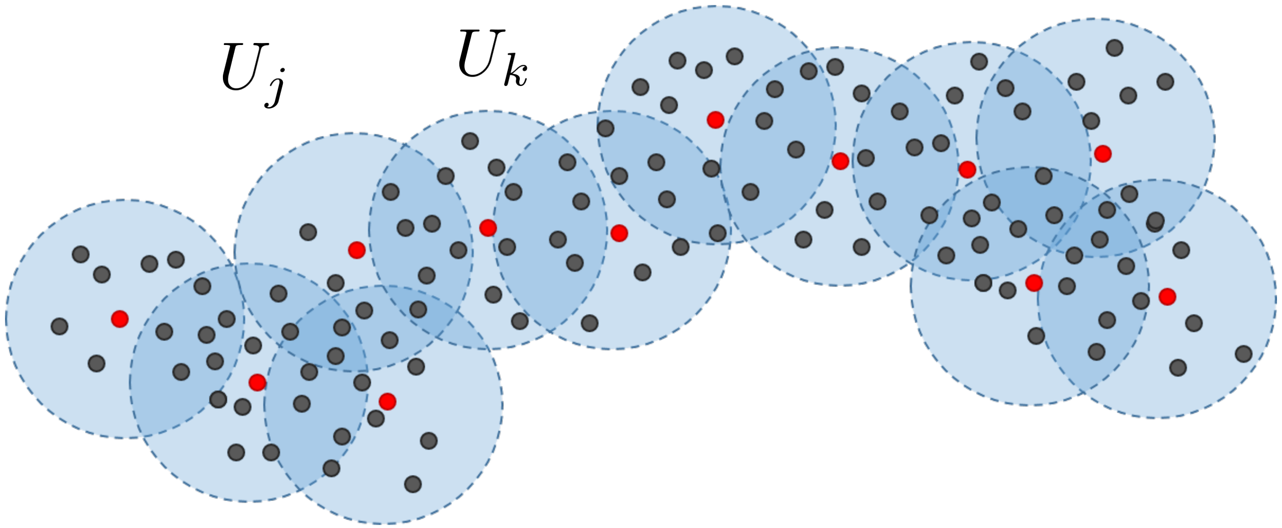

Partition of 1

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$\mathcal{N}(\mathcal{U})$

$$\check{H}^1(\mathcal{U};\mathscr{C}_G) \longrightarrow \check{H}^1(B;\mathscr{C}_G) \xleftarrow{\cong} \mathsf{Prin}_G(B) \xleftarrow{\cong} [B, BG]$$

$$[\{f_{jk}\}] \qquad\qquad [f]$$

$$f : B \longrightarrow BG$$

$$\mathcal{U} = \{U_j\}_{j\in J}$$

Partition of 1

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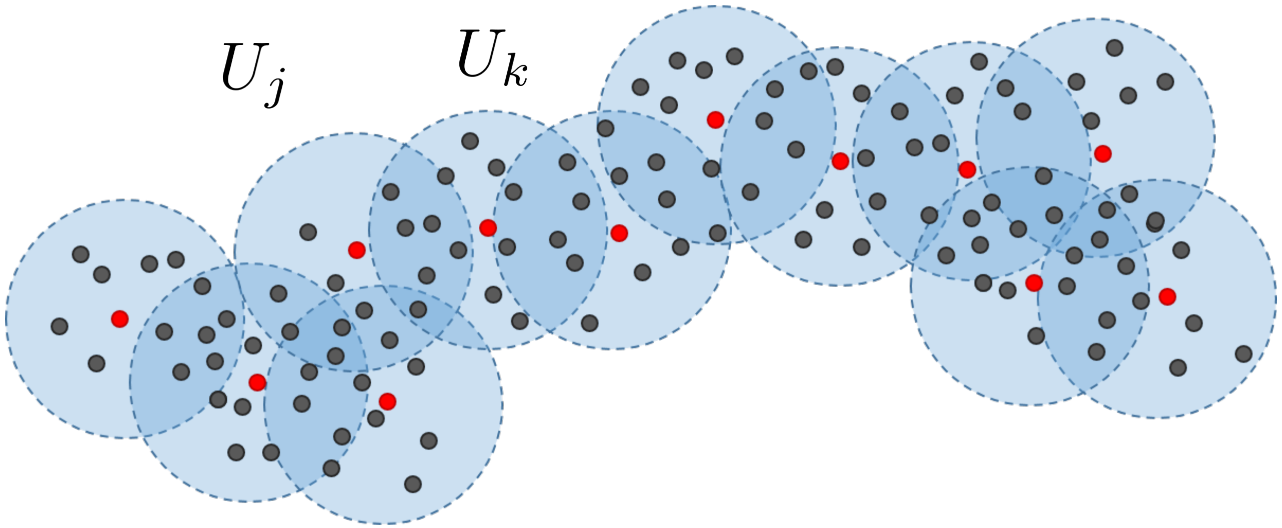

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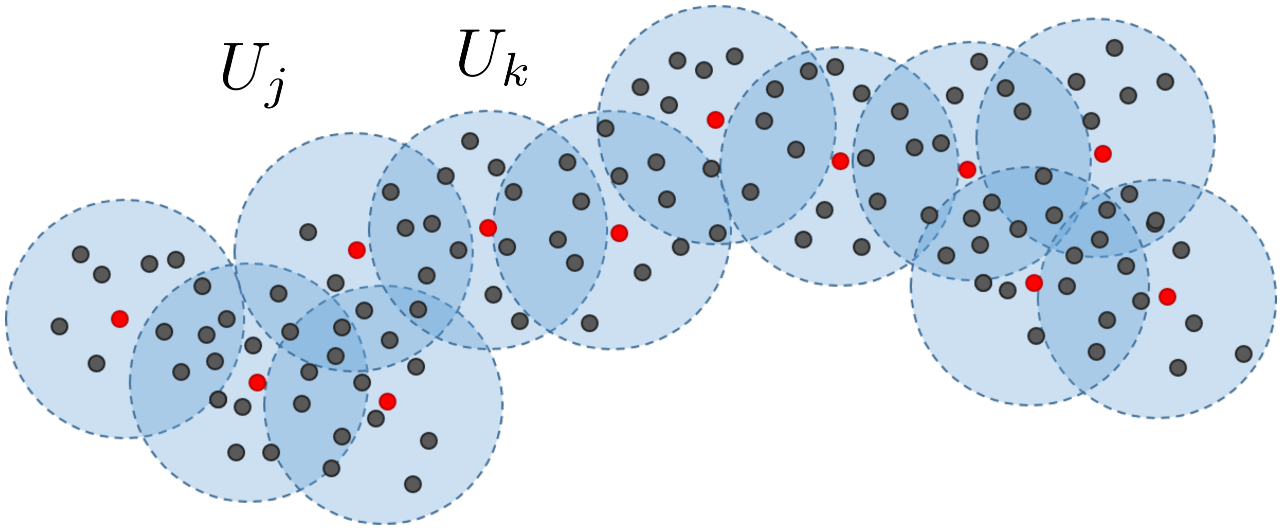
$$[\{f_{jk}\}] \qquad\qquad [f]$$

(Milnor)

$$f : B \longrightarrow BG = (G * G * \cdots)/G$$

$$\mathcal{U} = \{U_j\}_{j\in J}$$



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$[\{f_{jk}\}]$ $\qquad$ $[f]$

(Milnor)

$$f : B \longrightarrow BG = (G * G * \cdots)/G$$

$$U_j \ni b \mapsto \Big[\varphi_0(b), f_{j0}(b) : \varphi_1(b), f_{j1}(b) : \cdots\Big]$$

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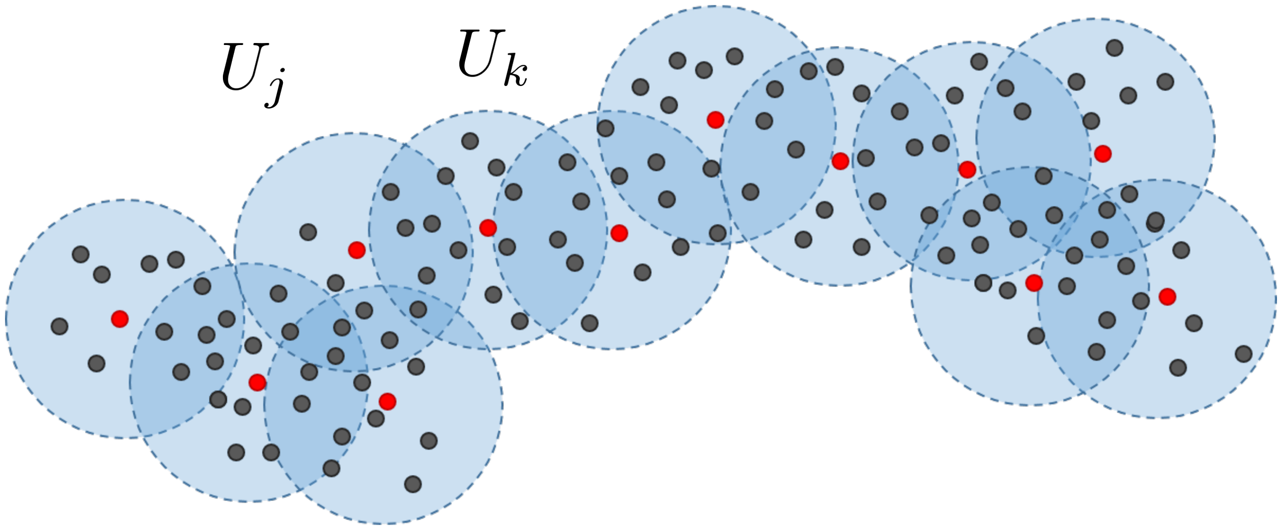
Partition of 1

$$\{\varphi_j\}_{j\in J}$$



$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

Circular Coordinates:

G	$\mathbb{Z}$
BG	S^1

$$\mathcal{U} = \{U_j\}_{j \in J}$$

Partition of 1

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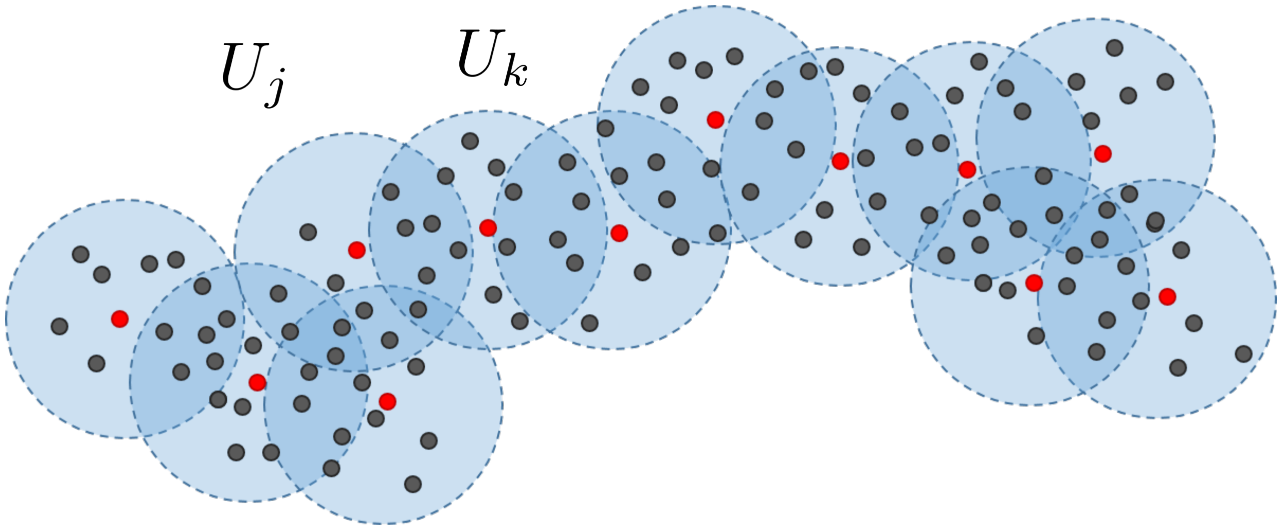

$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

Circular Coordinates: $[\eta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z})$

G	$\mathbb{Z}$
BG	S^1

$$\mathcal{U} = \{U_j\}_{j \in J}$$

Partition of 1

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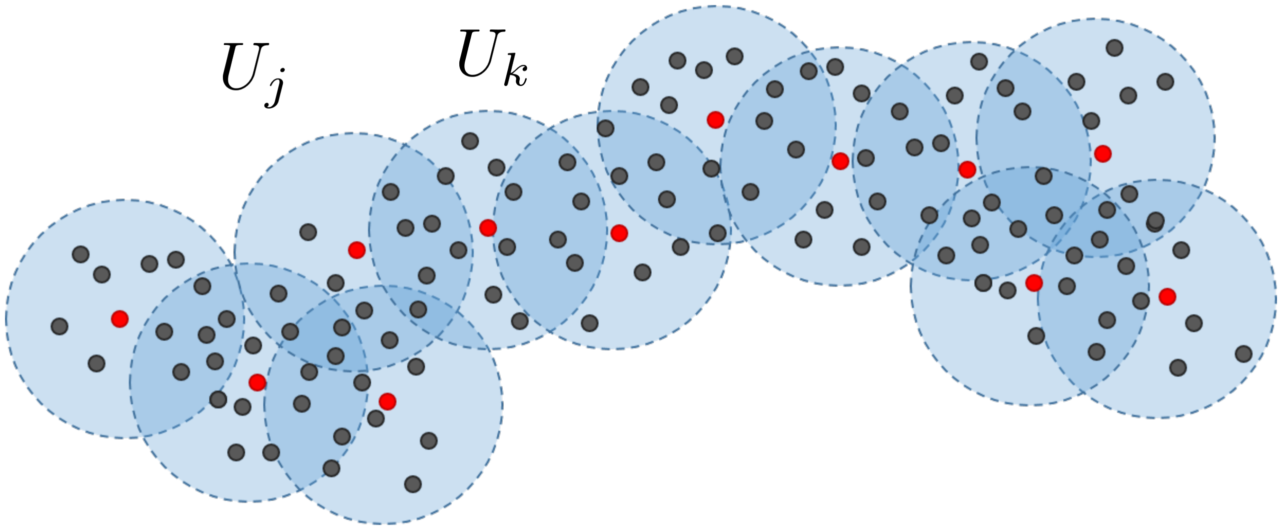

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Circular Coordinates: $[\eta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z})$

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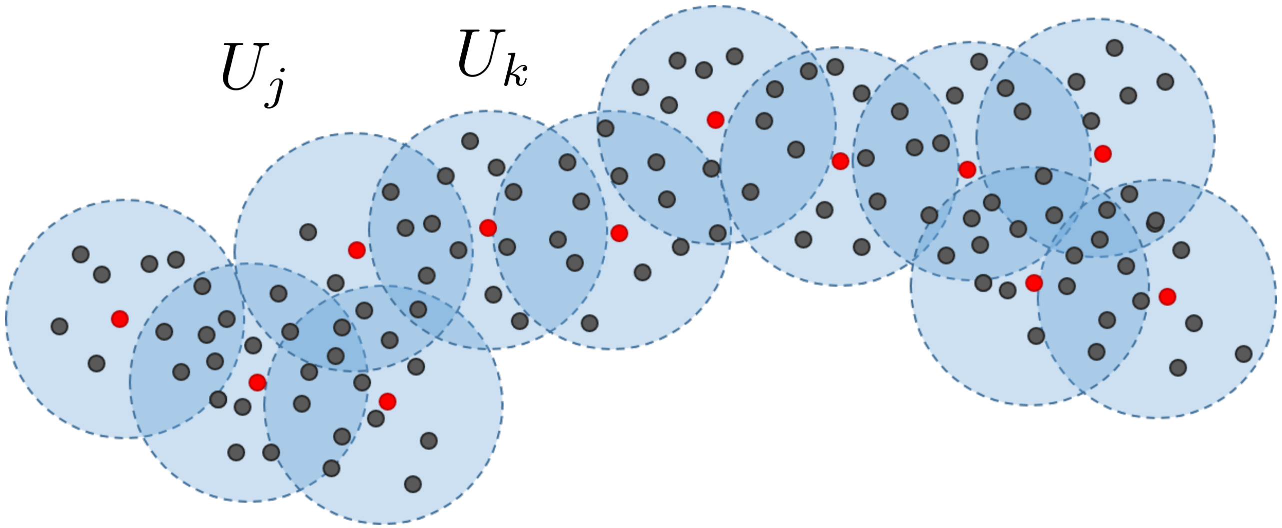


$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

Circular Coordinates:

$[\eta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z})$ → Harmonic Cocycle $\theta = \eta + \delta^0 \tau$

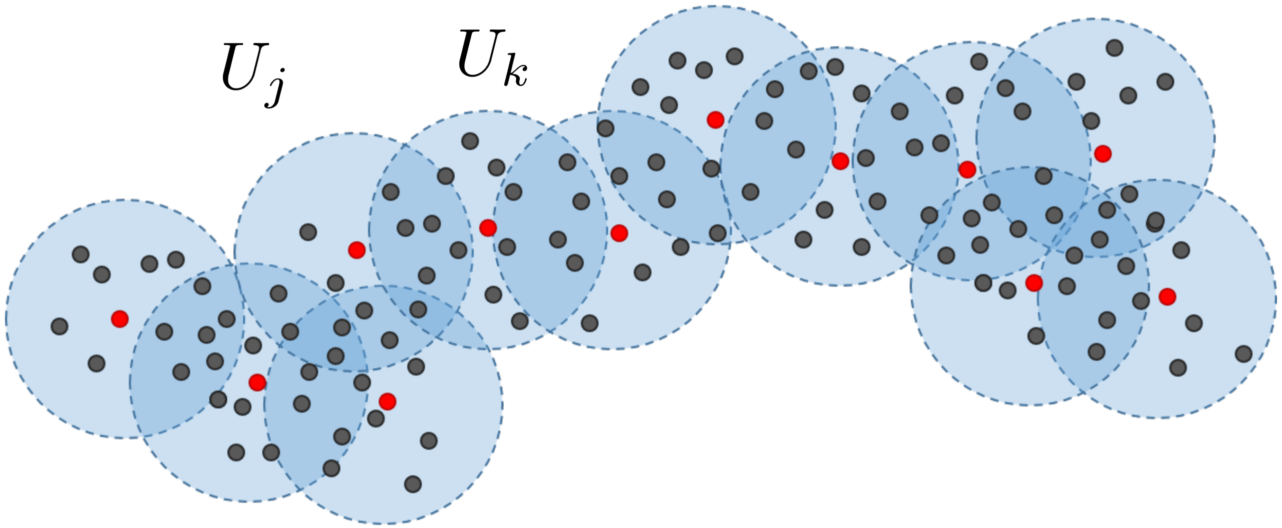
G	$\mathbb{Z}$
BG	S^1

$$f : B \longrightarrow S^1$$

$$\mathcal{U} = \{U_j\}_{j \in J}$$

Partition of 1

$$\{\varphi_j\}_{j \in J}$$



$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

Harmonic Cocycle

Circular Coordinates:

$$[\eta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}) \longrightarrow \theta = \eta + \delta^0 \tau$$

G	$\mathbb{Z}$
BG	S^1

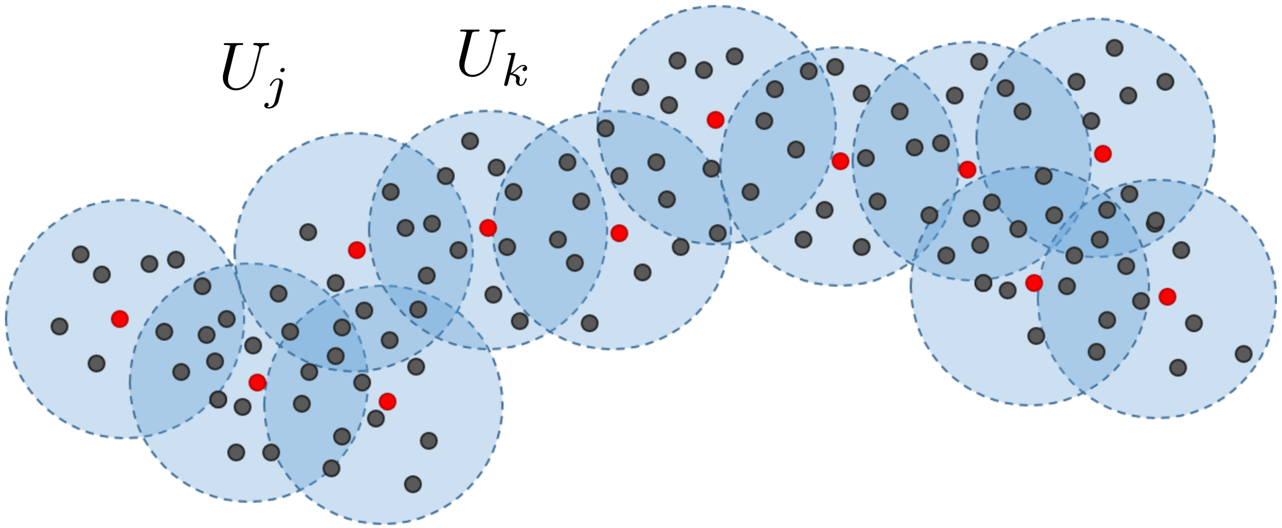
$$f : B \longrightarrow S^1$$

$$U_j \ni b \mapsto \exp\left\{2\pi i\left(\tau_j + \sum_k \varphi_k(b)\theta_{jk}\right)\right\}$$

$$\mathcal{U} = \{U_j\}_{j \in J}$$

Partition of 1

$$\{\varphi_j\}_{j\in J}$$



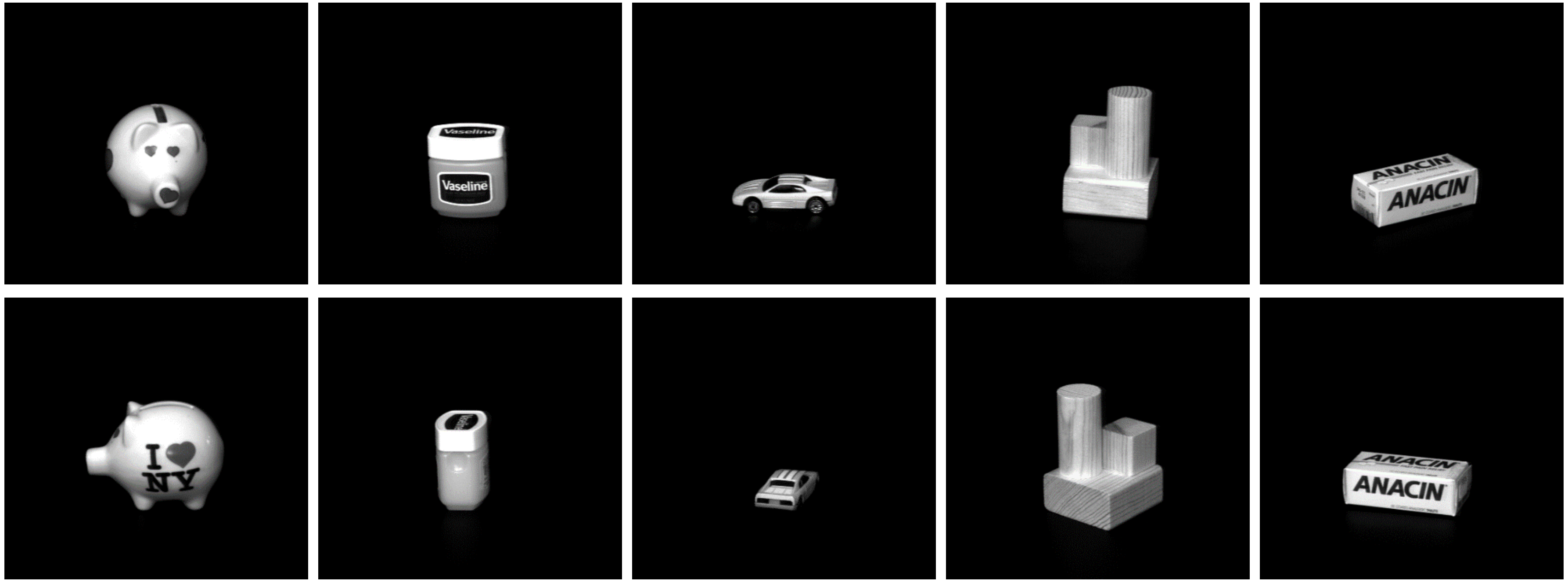
$$B = \bigcup \mathcal{U}$$

$$\mathcal{N}(\mathcal{U})$$

COIL-5 (Columbia Image Library)

(Unprocessed)

5 Objects



72 Rotations

Projective Coords ($\mathbb{R}$):

$G = \mathbb{Z}/2 = \{\pm 1\}$	$B\mathbb{Z}/2 = \mathbb{R}\mathbf{P}^{\infty}$

$$\mathcal{U} = \{U_j\}_{j \in J}$$

$$|J| = n + 1$$

Projective Coords ($\mathbb{R}$):

$$G = \mathbb{Z}/2 = \{\pm 1\}$$

$$B\mathbb{Z}/2 = \mathbb{R}\mathbf{P}^\infty$$

$$\mathcal{U} = \{U_j\}_{j \in J}$$

$$|J| = n + 1$$

$$[\theta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}/2)$$

$$f : B \longrightarrow \mathbb{R}\mathbf{P}^n$$

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$G = \mathbb{Z}/2 = \{\pm 1\}$	$B\mathbb{Z}/2 = \mathbb{R}\mathbf{P}^\infty$

$$[\theta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}/2)$$

$$\mathcal{U} = \{U_j\}_{j\in J}$$

$$|J| = n + 1$$

$$f : B \longrightarrow \mathbb{R}\mathbf{P}^n$$

$$U_j \ni b \mapsto \left[\theta_{j0}\sqrt{\varphi_0(b)} : \cdots : \theta_{jn}\sqrt{\varphi_n(b)}\right]$$

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$G = \mathbb{Z}/2 = \{\pm 1\}$	$B\mathbb{Z}/2 = \mathbb{R}\mathbf{P}^\infty$

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$$\mathcal{U} = \{U_j\}_{j \in J}$$

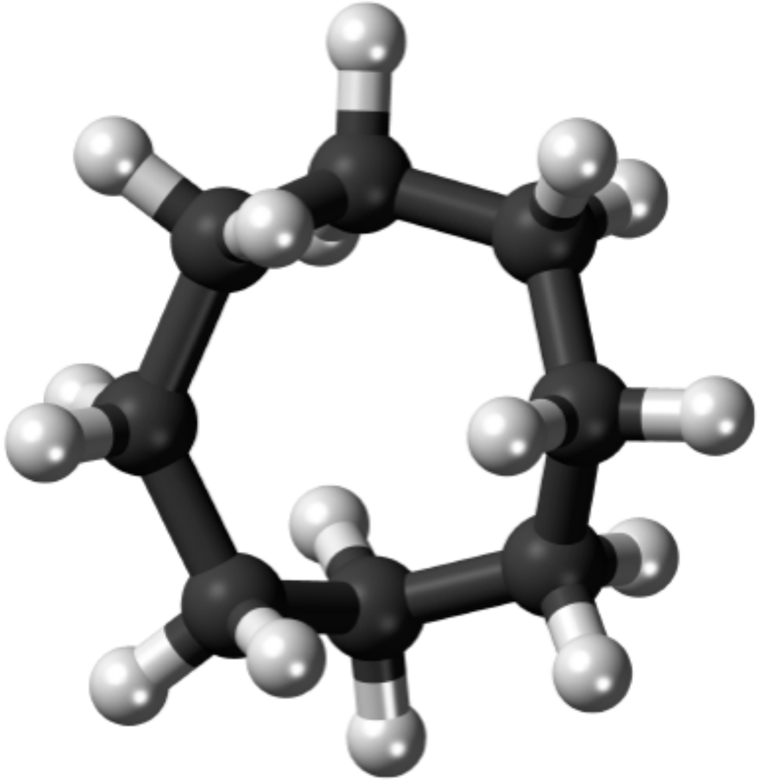
$$|J| = n + 1$$

$$f : B \longrightarrow \mathbb{R}\mathbf{P}^n$$

$$U_j \ni b \mapsto \left[\theta_{j0}\sqrt{\varphi_0(b)} : \cdots : \theta_{jn}\sqrt{\varphi_n(b)}\right]$$

$\mathbb{R}\mathbf{P}^n$ Supports a version of Principal Component Analysis (Projective PCA)

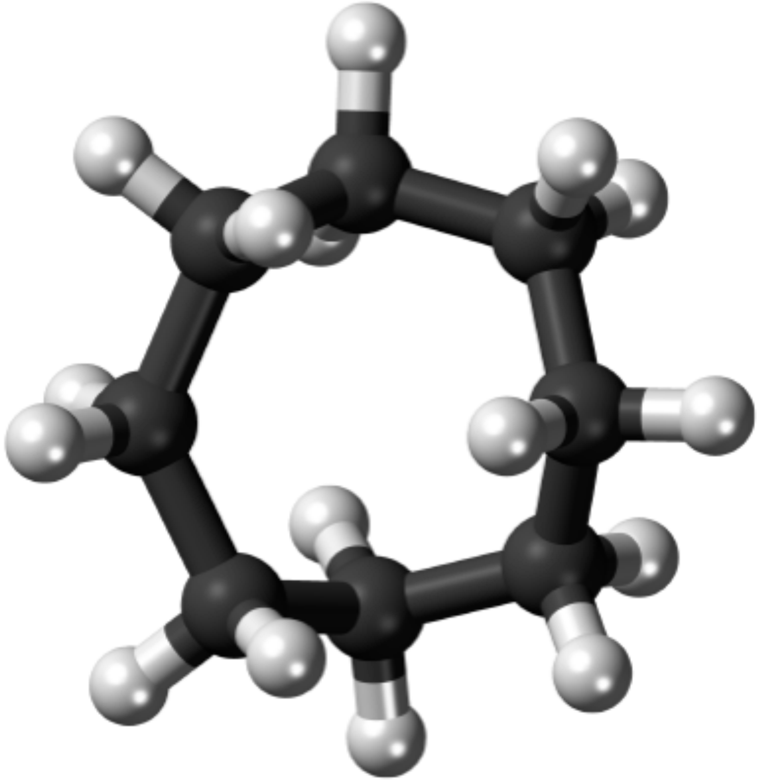
Conformation space of Cyclo-Octane



C_8H_{16}

Topology of Cyclo-Octane Energy Landscape, S. Martin et. al., **JCP**, 2010

Conformation space of Cyclo-Octane



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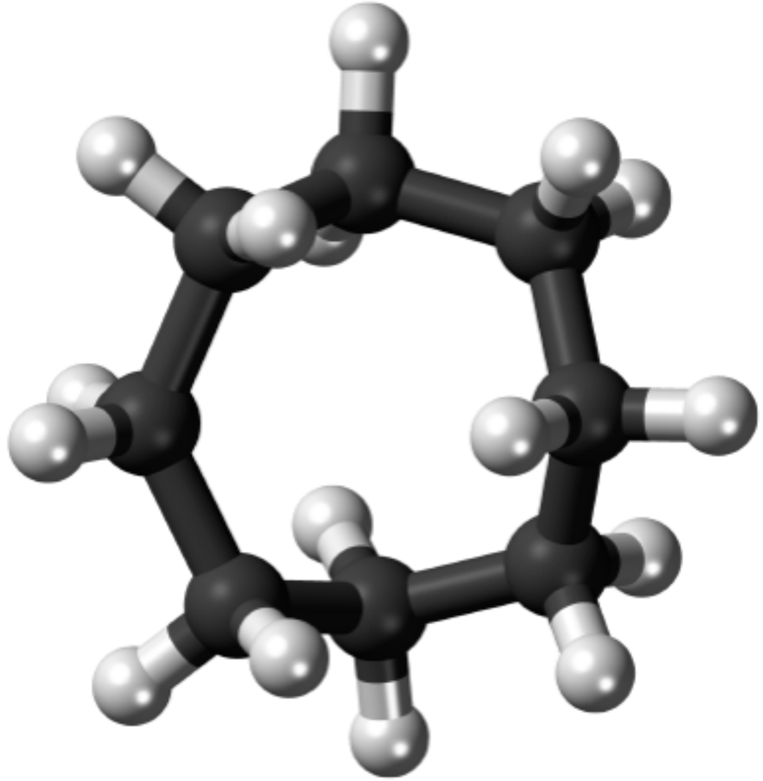
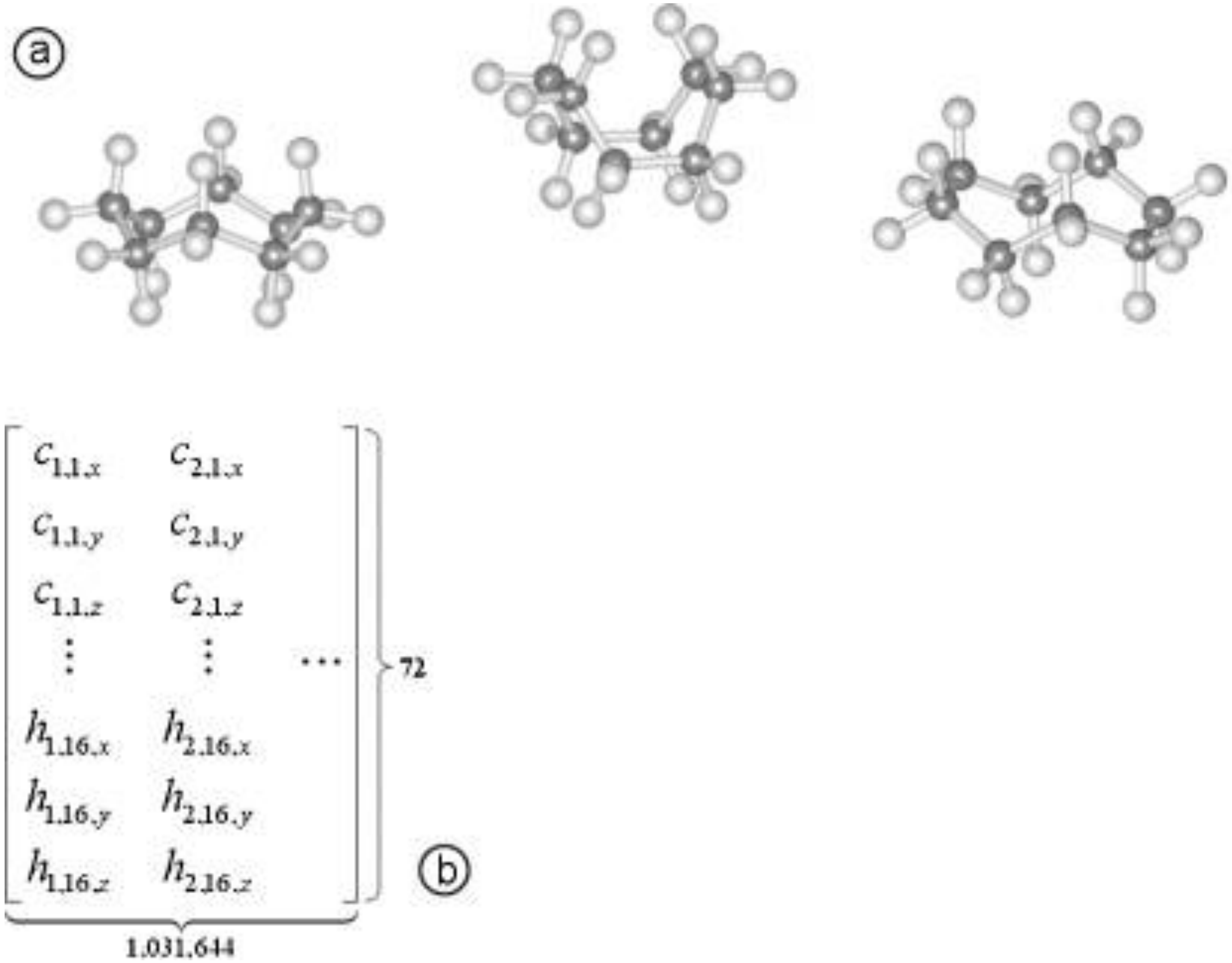
Conformation space of Cyclo-Octane

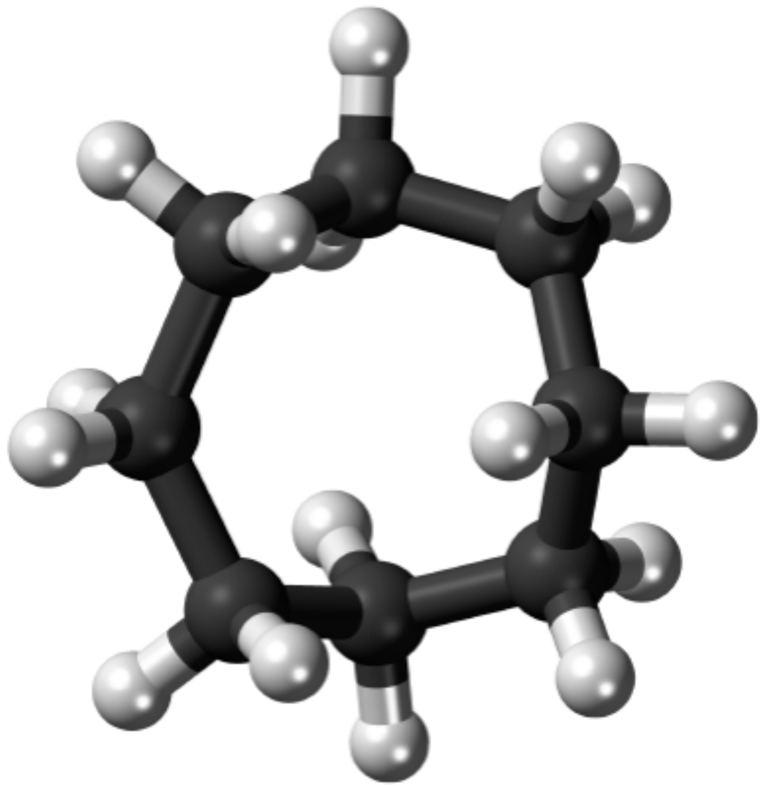


C_8H_{16}

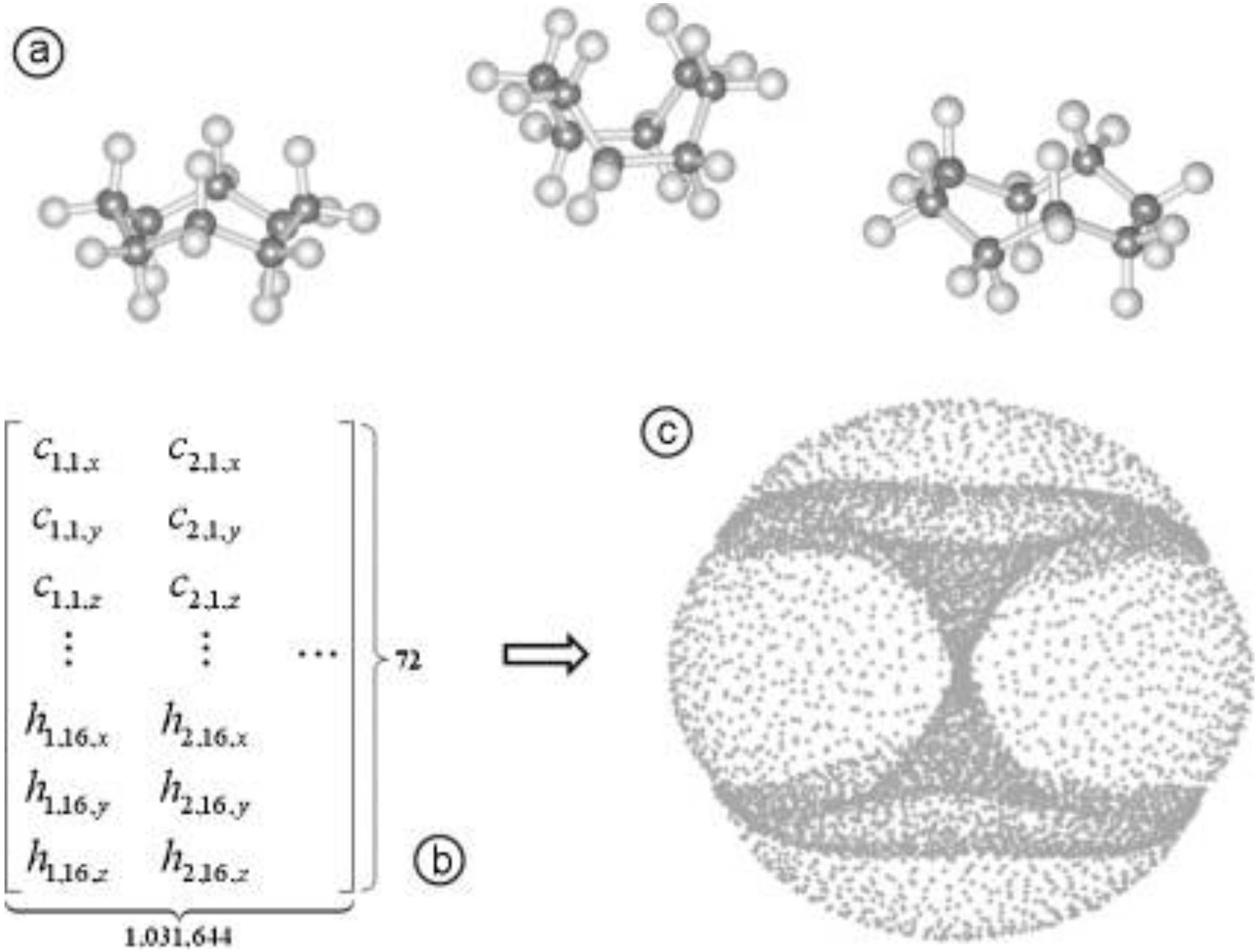


Topology of Cyclo-Octane Energy Landscape, S. Martin et. al., **JCP**, 2010

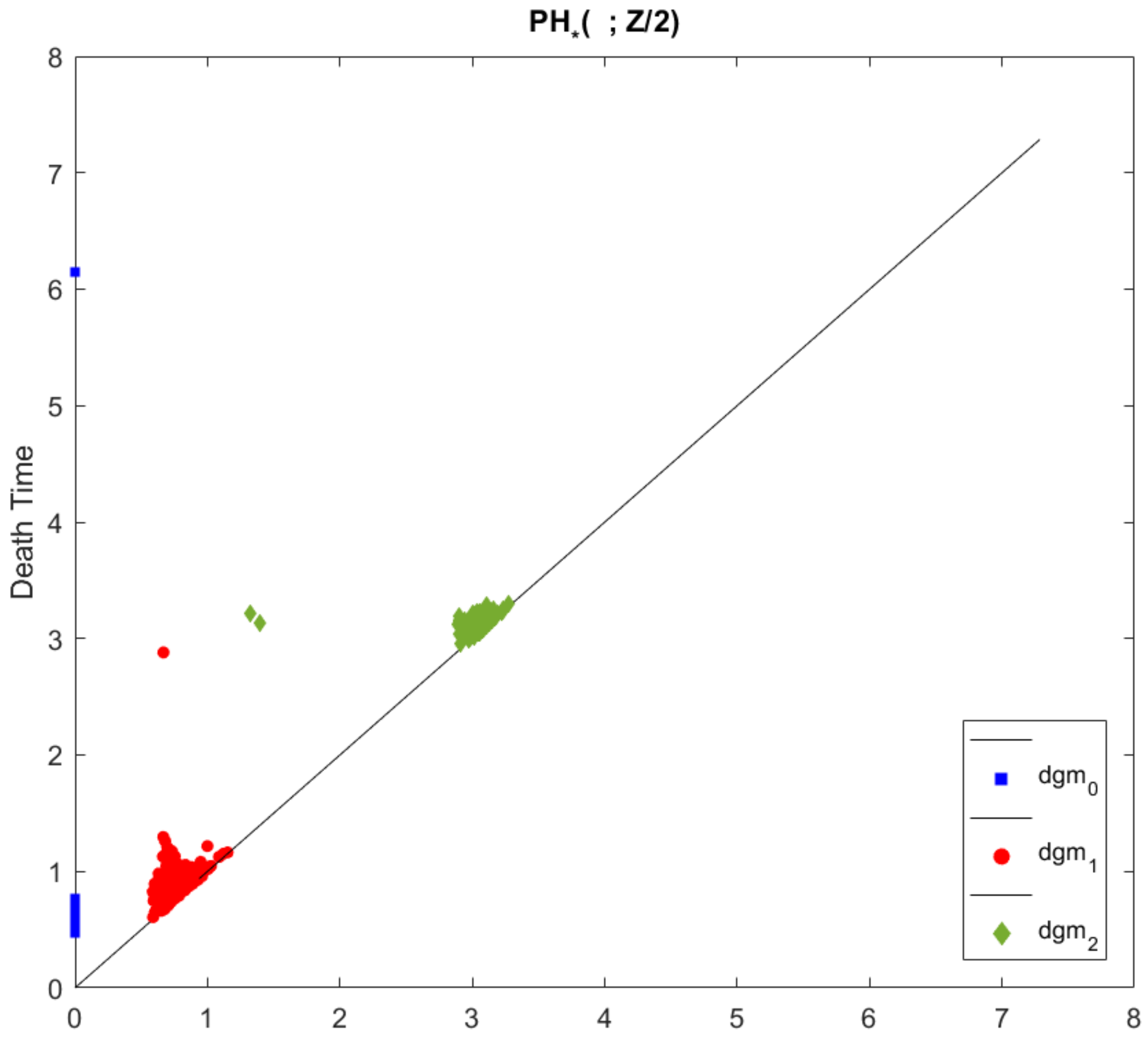
Conformation space of Cyclo-Octane

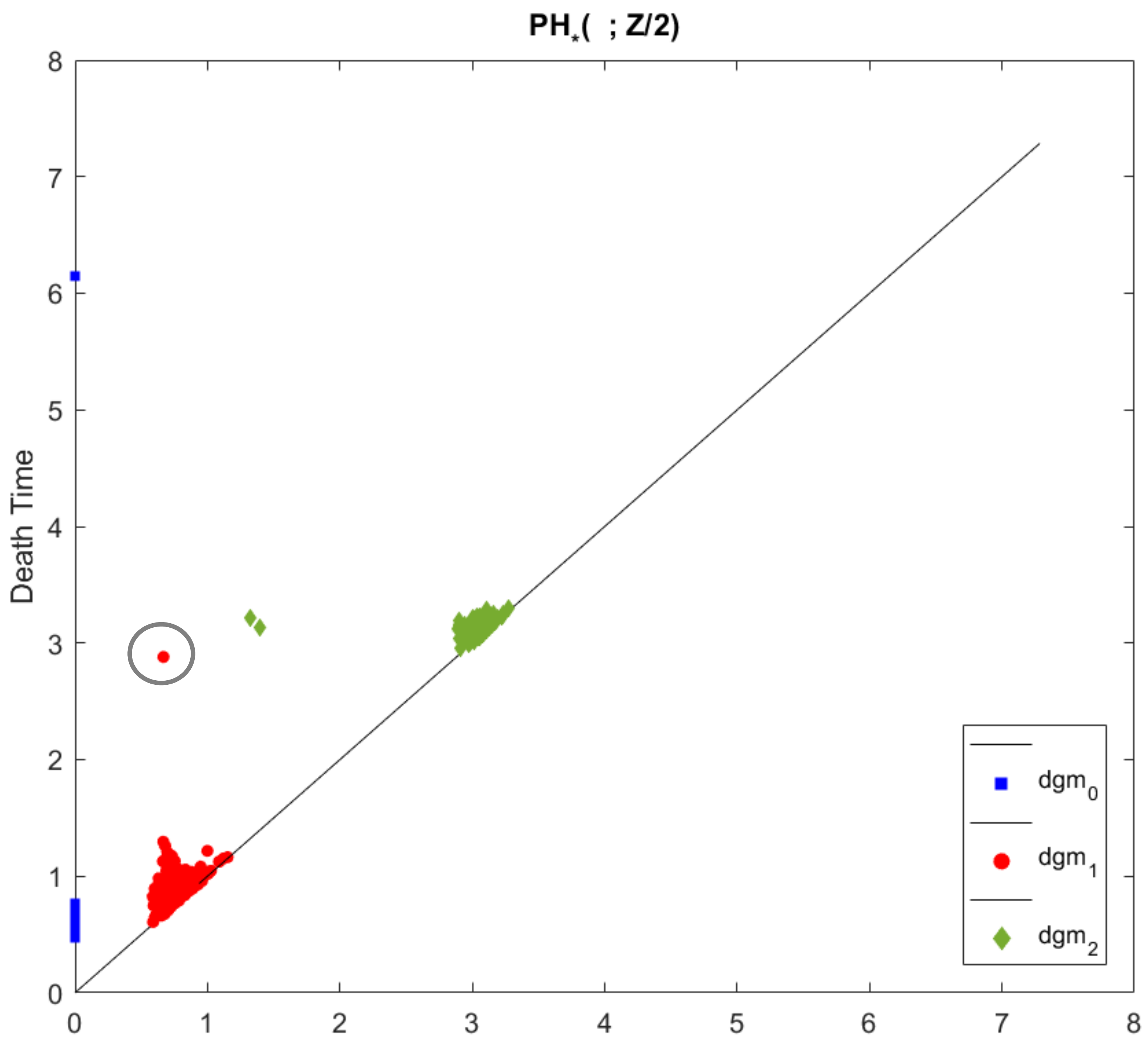


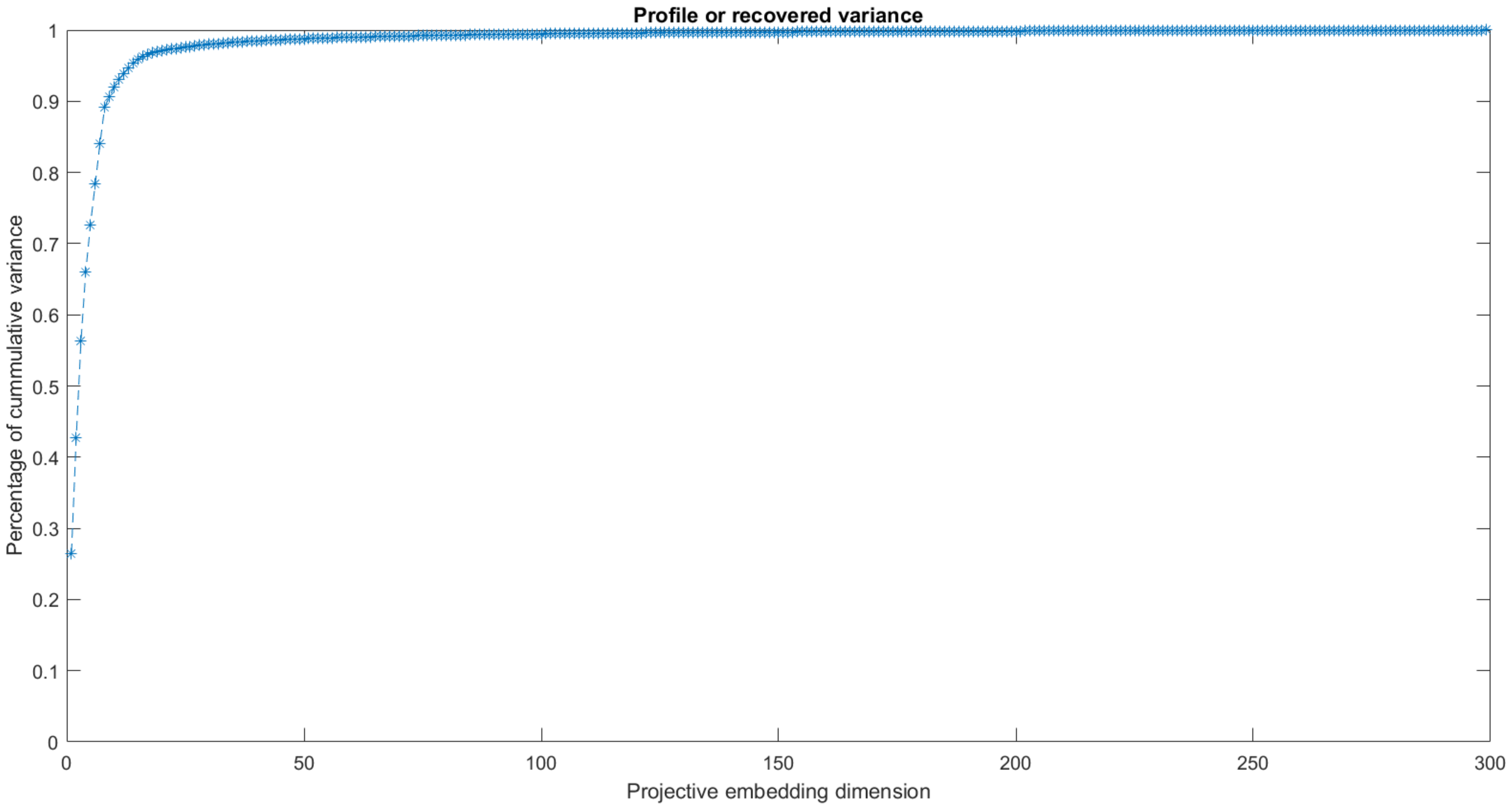
C_8H_{16}



Topology of Cyclo-Octane Energy Landscape, S. Martin et. al., **JCP**, 2010


PH*(; Z/2)
Death Time
0
1
2
3
4
5
6
7
8
dgm0
dgm1
dgm2


PH*(; Z/2)
Death Time
dgm0
dgm1
dgm2
0
1
2
3
4
5
6
7
8



Profile or recovered variance
Percentage of cummulative variance
Projective embedding dimension
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
50
100
150
200
250
300

The state of the art (BG-Coordinates):

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- Sparse, stable and transductive circular coordinates:

$$[\eta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}) \quad \Longrightarrow \quad f_\eta : B \longrightarrow S^1$$

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- Real Projective coordinates :

$$[\theta] \in H^1(\mathcal{N}(\mathcal{U}); \mathbb{Z}/2) \quad \Rightarrow \quad f_\theta : B \longrightarrow \mathbb{R}\mathbf{P}^k$$

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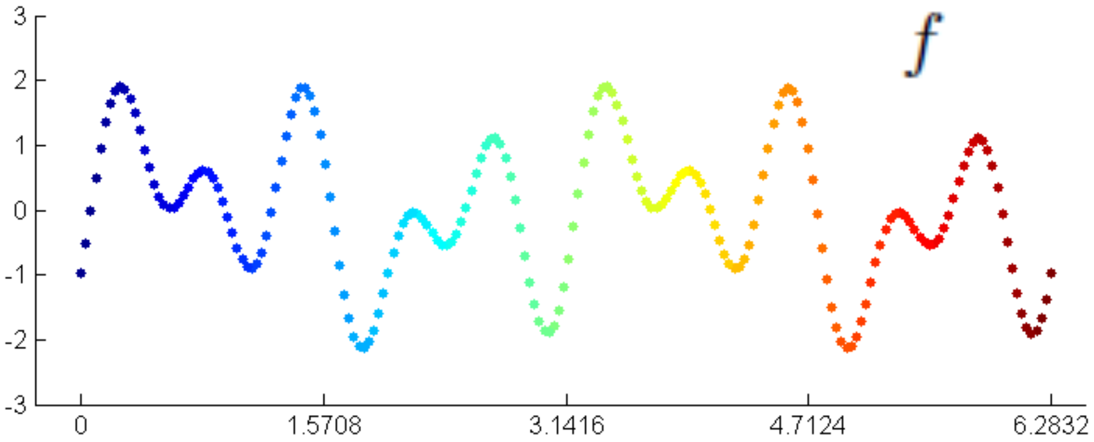
Reconstructing Dynamics

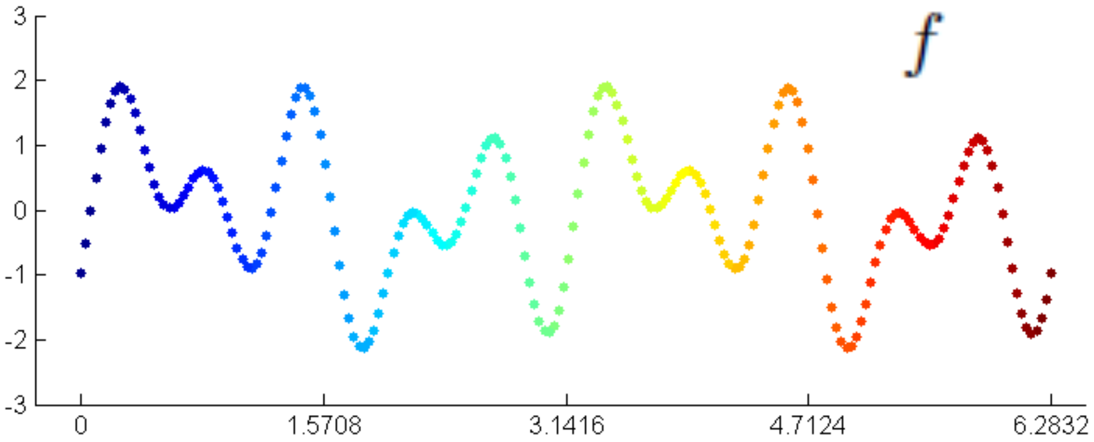
(Sliding Windows + Persistence + BG-Coordinates)

Sliding window embedding



Sliding Windows and Persistence: An application of topology to signal analysis, J. Perea and J. Harer, **FOCM** 2015

Sliding window embedding



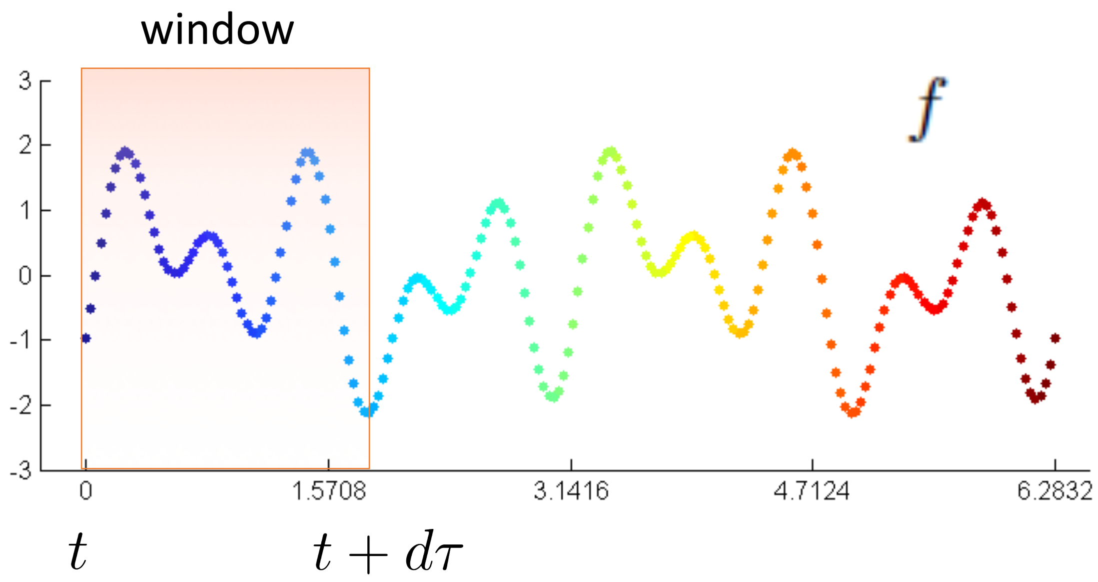
τ → Step/delay

$d\tau$ → Window size

$d + 1$ → Dimension

Sliding Windows and Persistence: An application of topology to signal analysis, J. Perea and J. Harer, **FOCM** 2015

Sliding window embedding



τ → Step/delay

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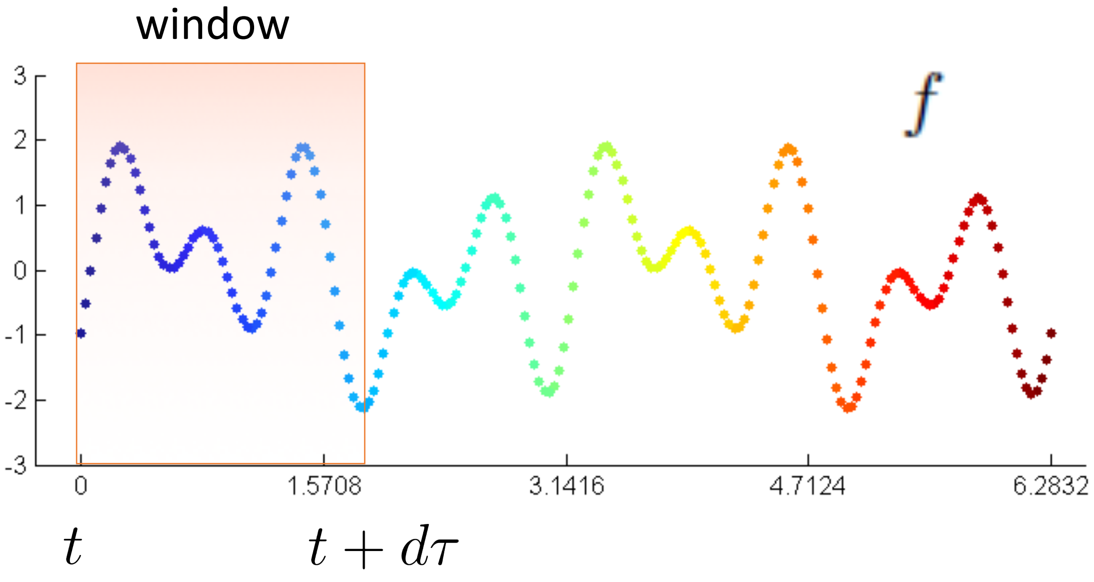
Sliding window embedding



τ → Step/delay

$d\tau$ → Window size

$d + 1$ → Dimension

$$SW_{d,\tau} f(t) = \begin{bmatrix} f(t) \\ f(t+\tau) \\ \vdots \\ f(t+d\tau) \end{bmatrix}$$

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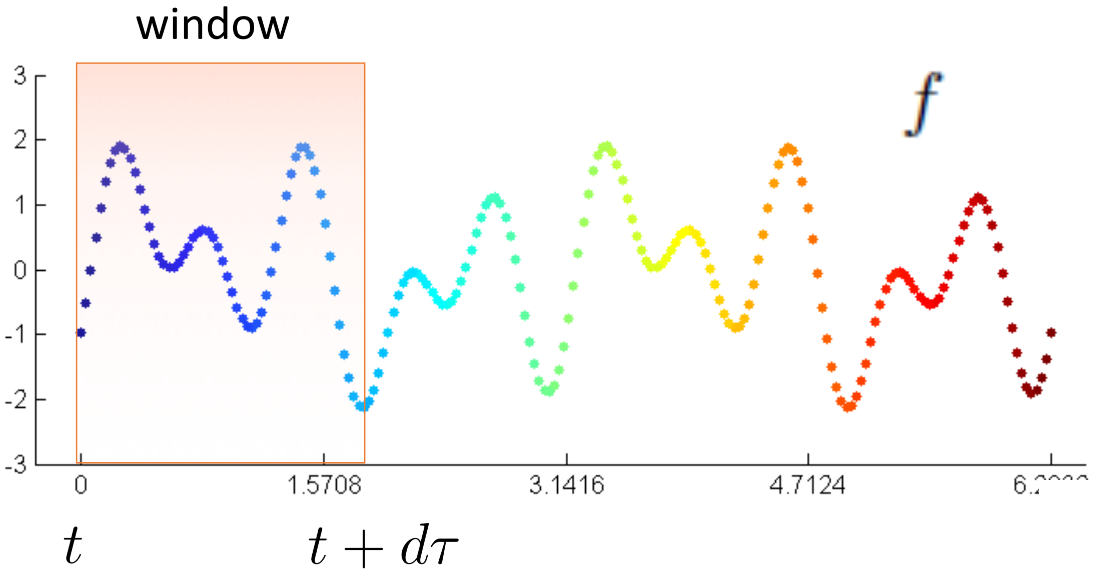
Sliding window embedding



τ → Step/delay

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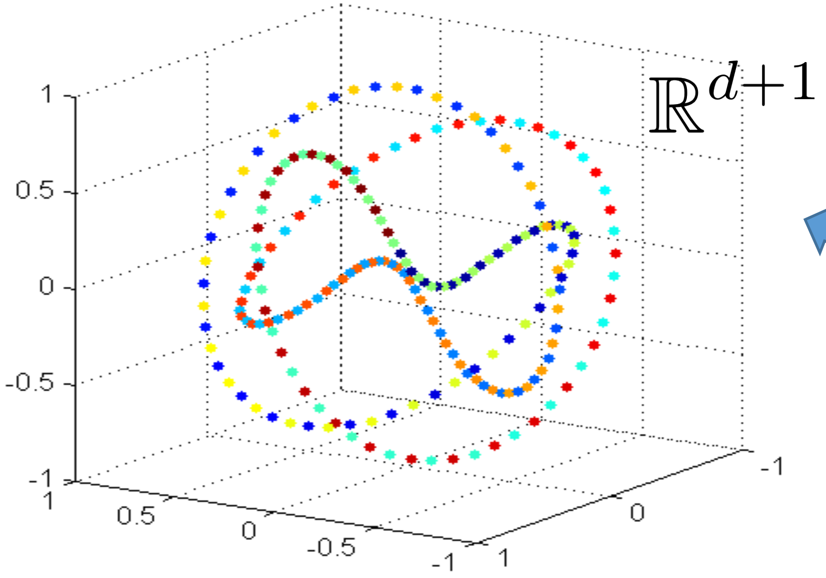


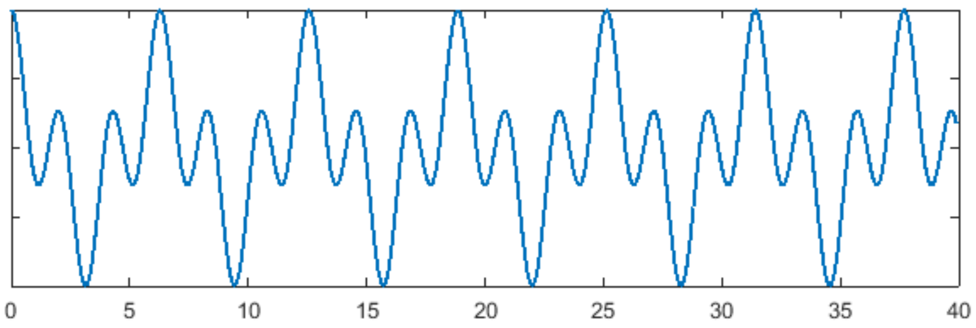
Sliding window point-cloud

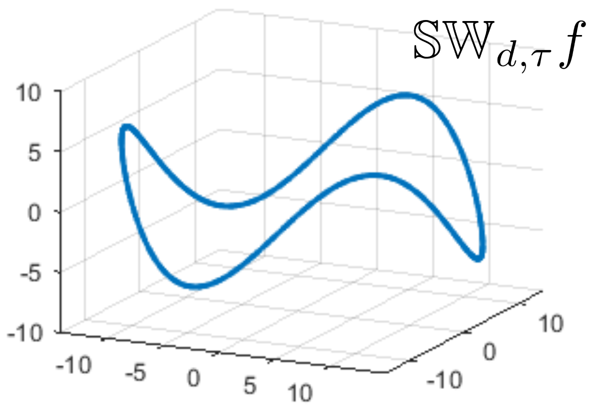
$$\mathbb{SW}_{d,\tau} f$$
$$||$$
$$SW_{d,\tau} f(I)$$

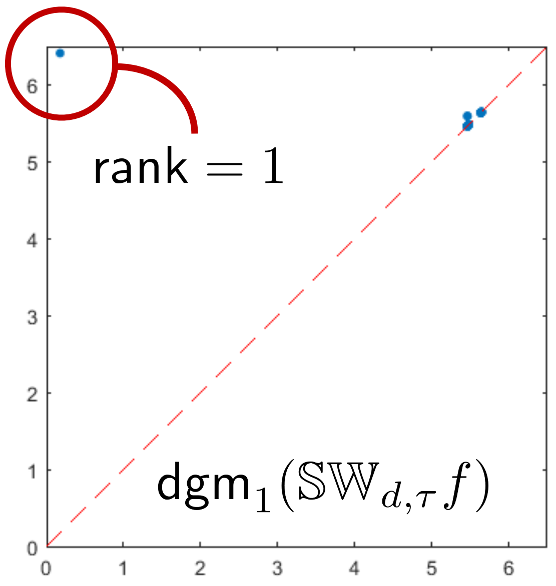
$I \subset \mathbb{R}$

Sliding Windows and Persistence: An application of topology to signal analysis, J. Perea and J. Harer, **FOCM** 2015

$$f(t) = \cos(t) + \cos(3t)$$





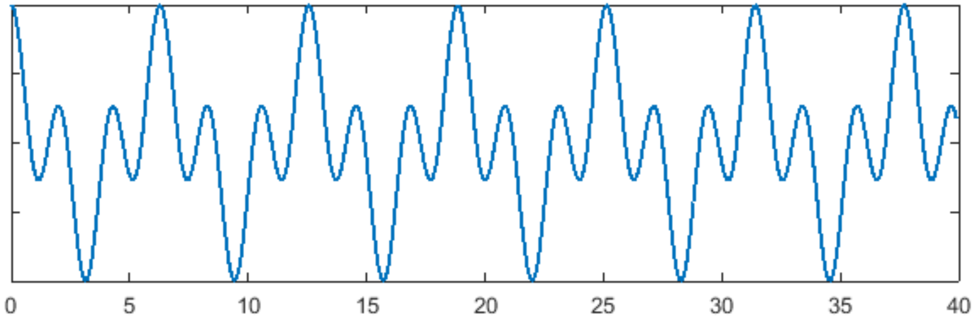


Time Series

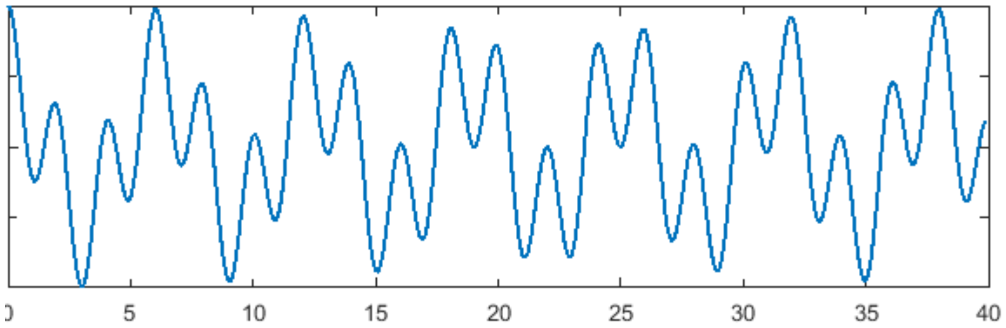
Sliding Window Point Cloud

Persistent Homology

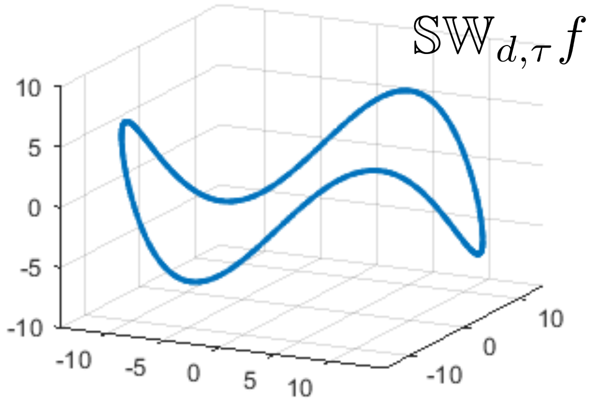
$$f(t) = \cos(t) + \cos(3t)$$



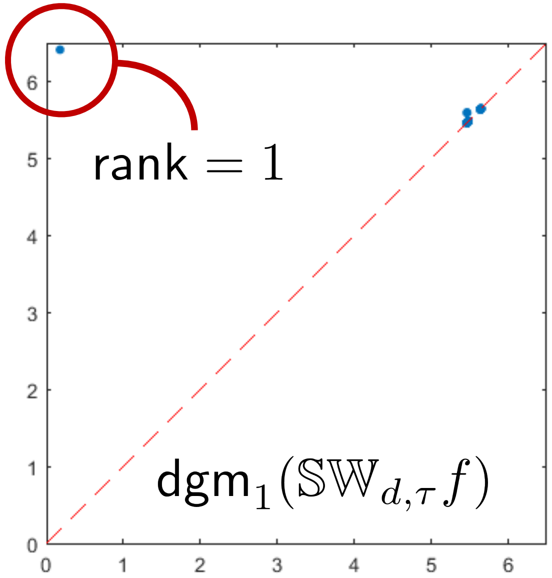
$$g(t) = \cos(t) + \cos(\pi t)$$



Time Series



Sliding Window Point Cloud



Persistent Homology

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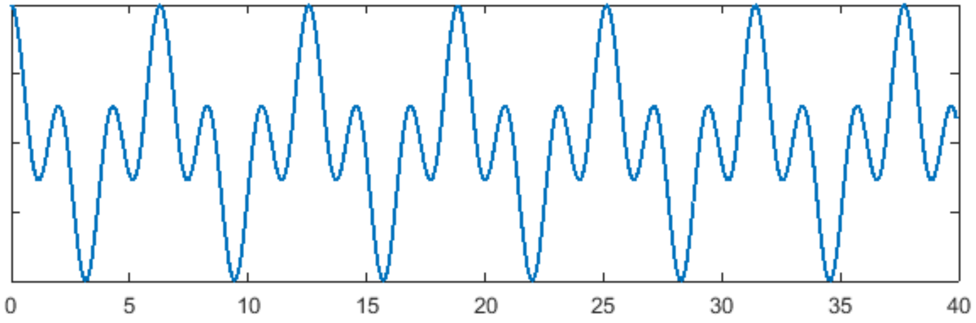

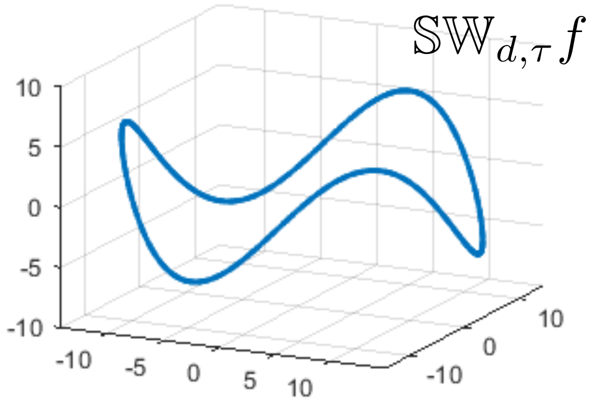

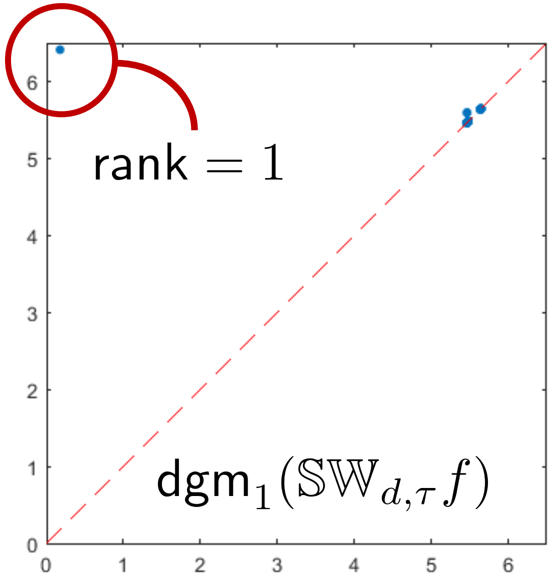

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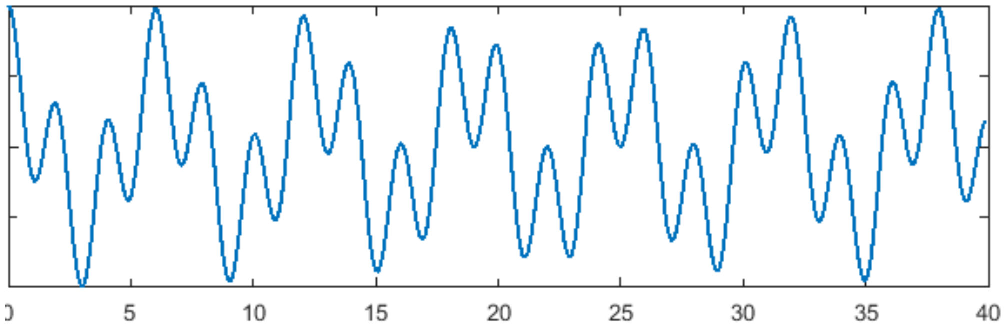

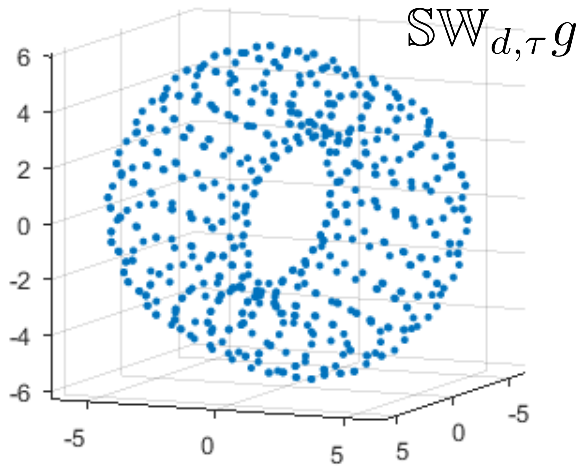

Sliding Window Point Cloud

Time Series

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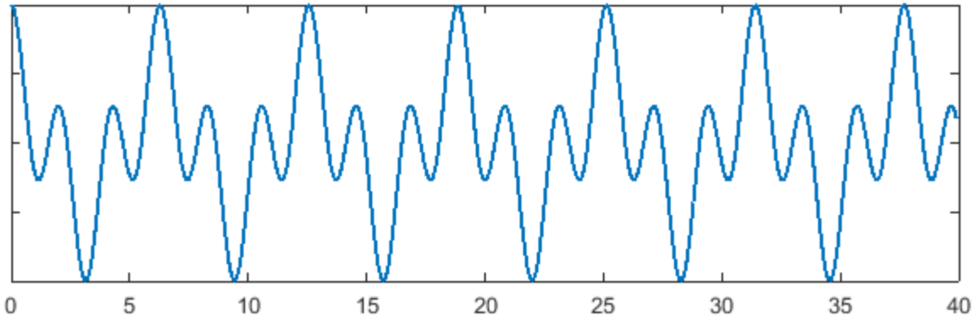

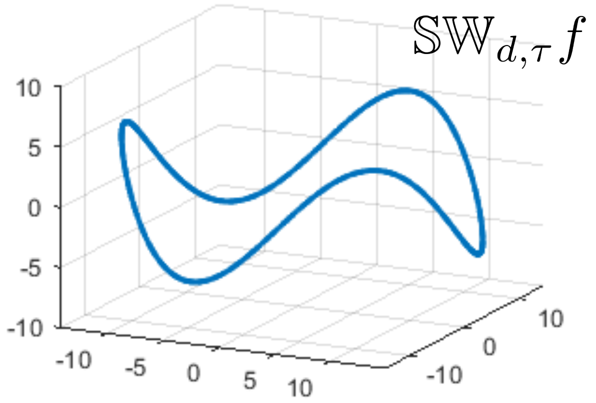

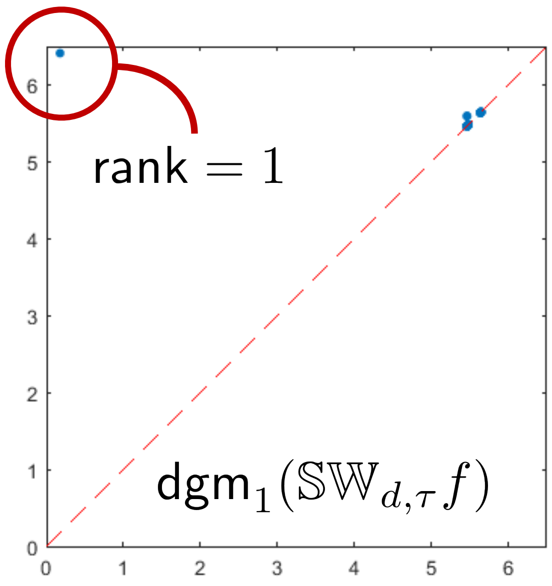

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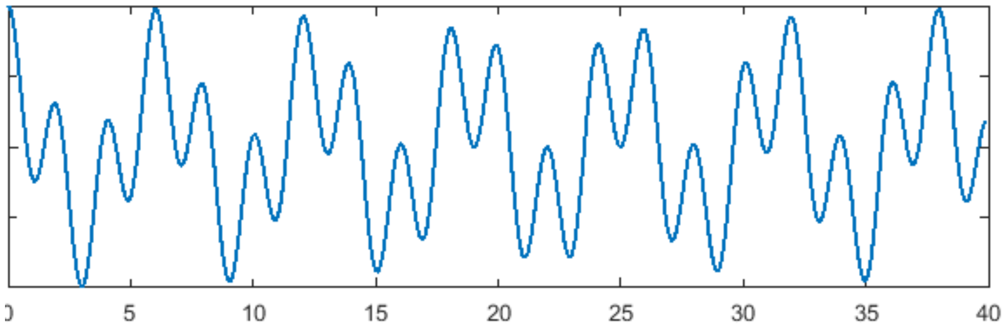
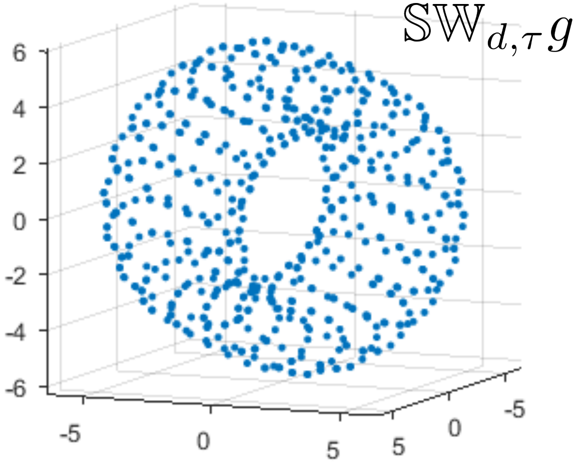
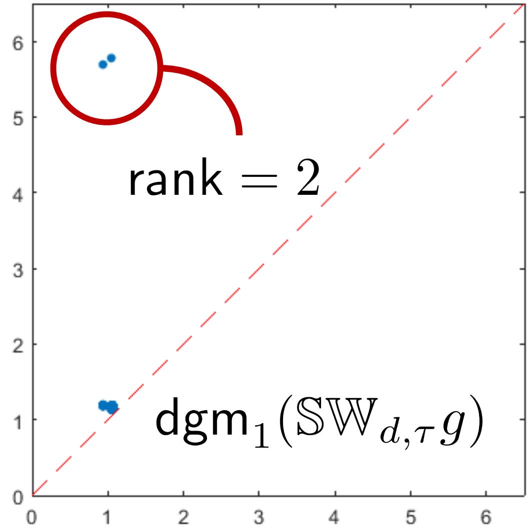


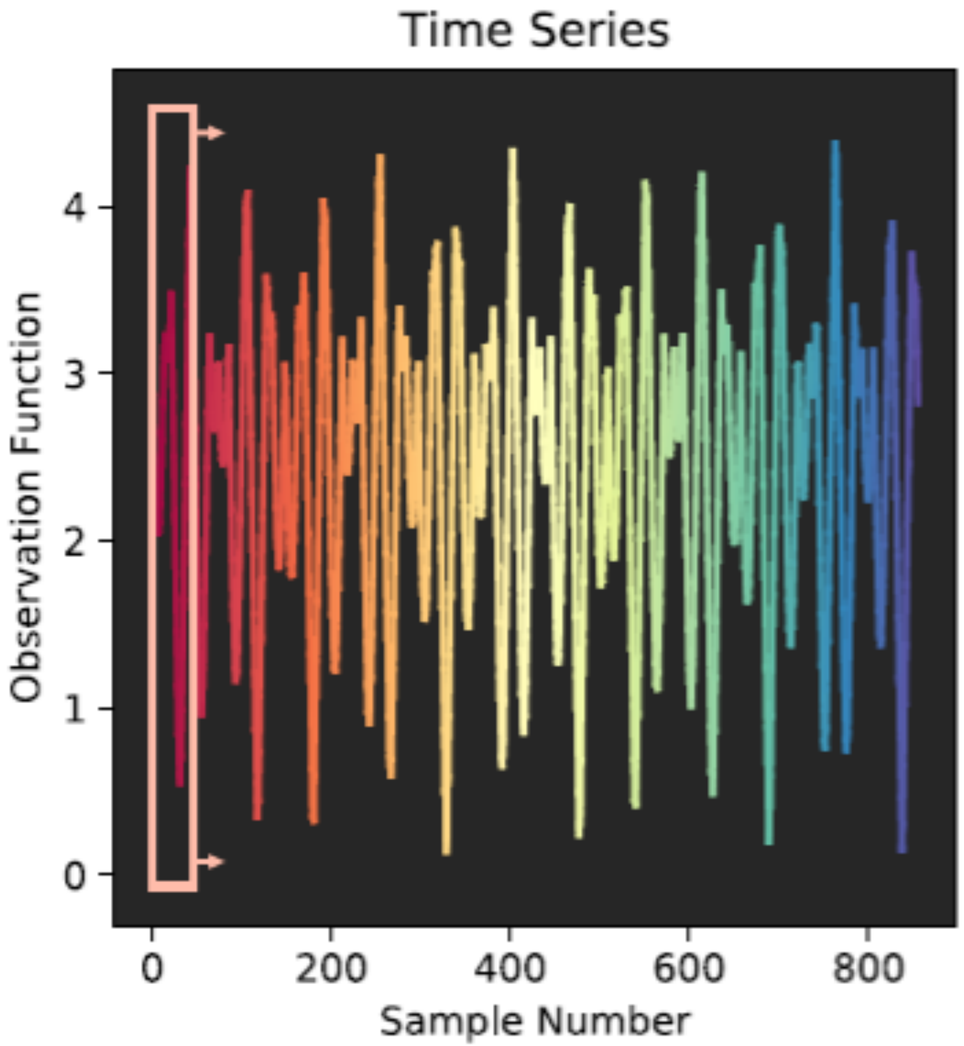




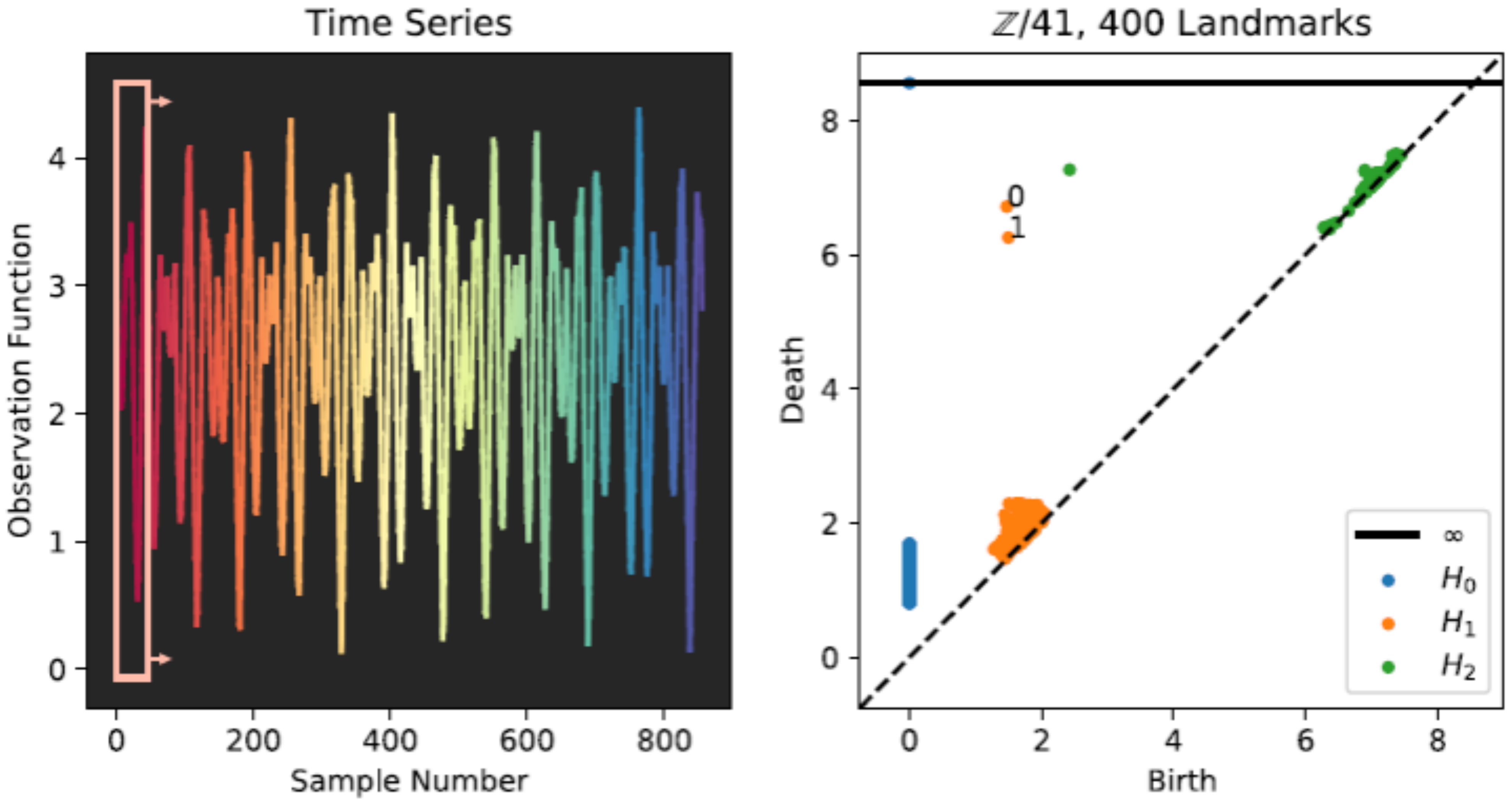
Time Series

Sliding Window Point Cloud

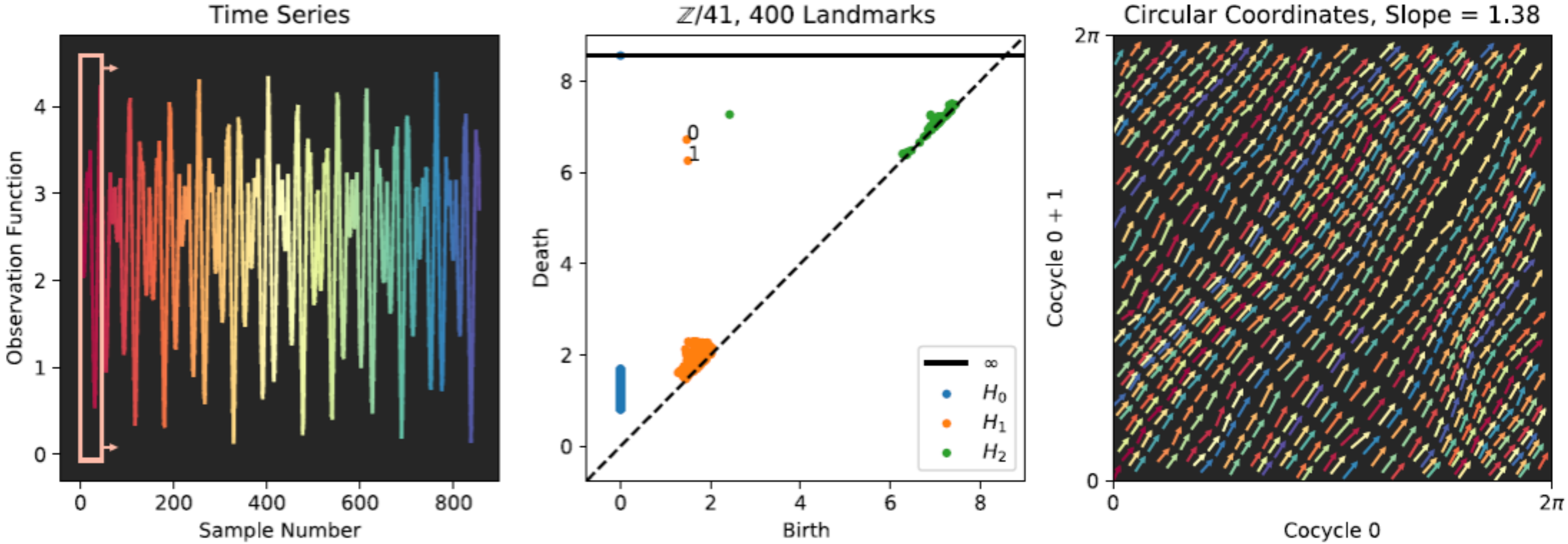
Persistent Homology


Time Series
Observation Function
Sample Number
0
1
2
3
4
0
200
400
600
800

Persistent Homology of Sliding Window Point Cloud

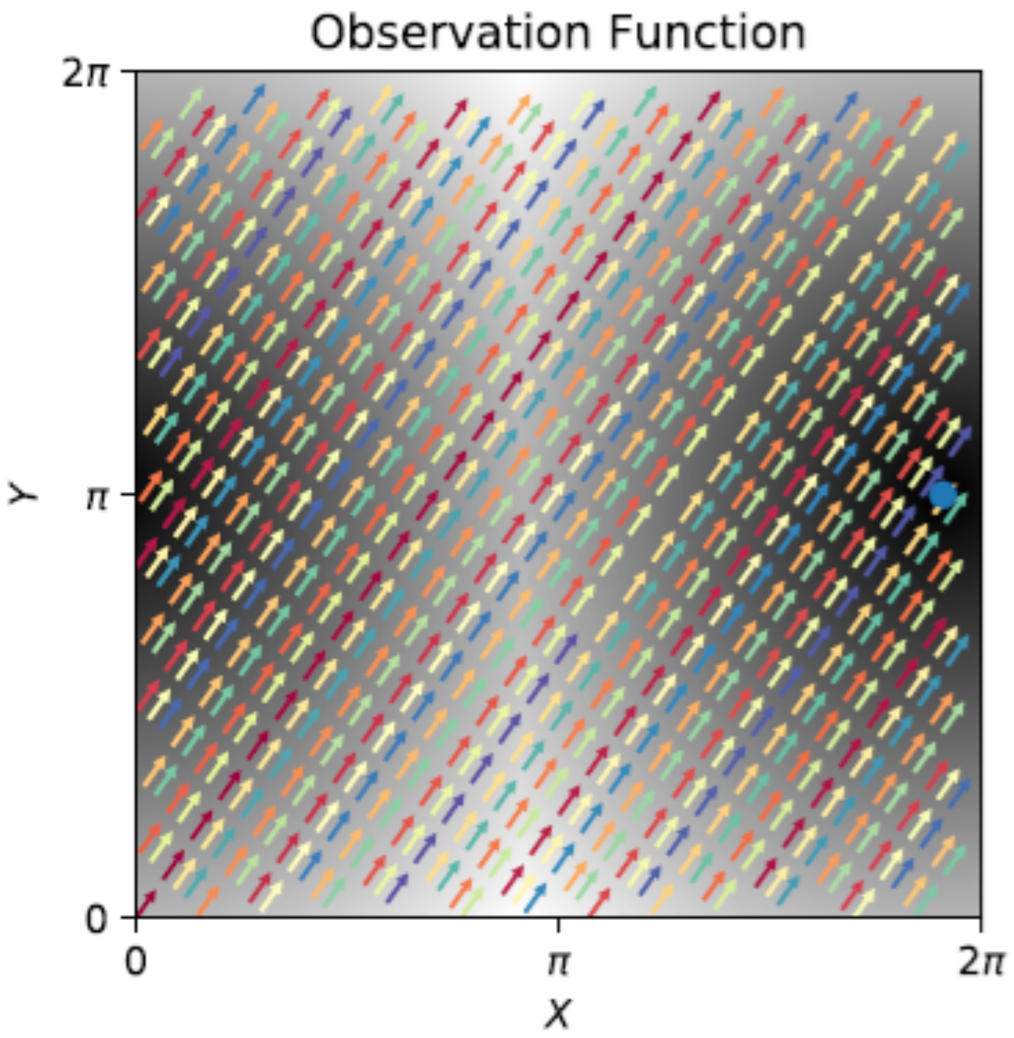


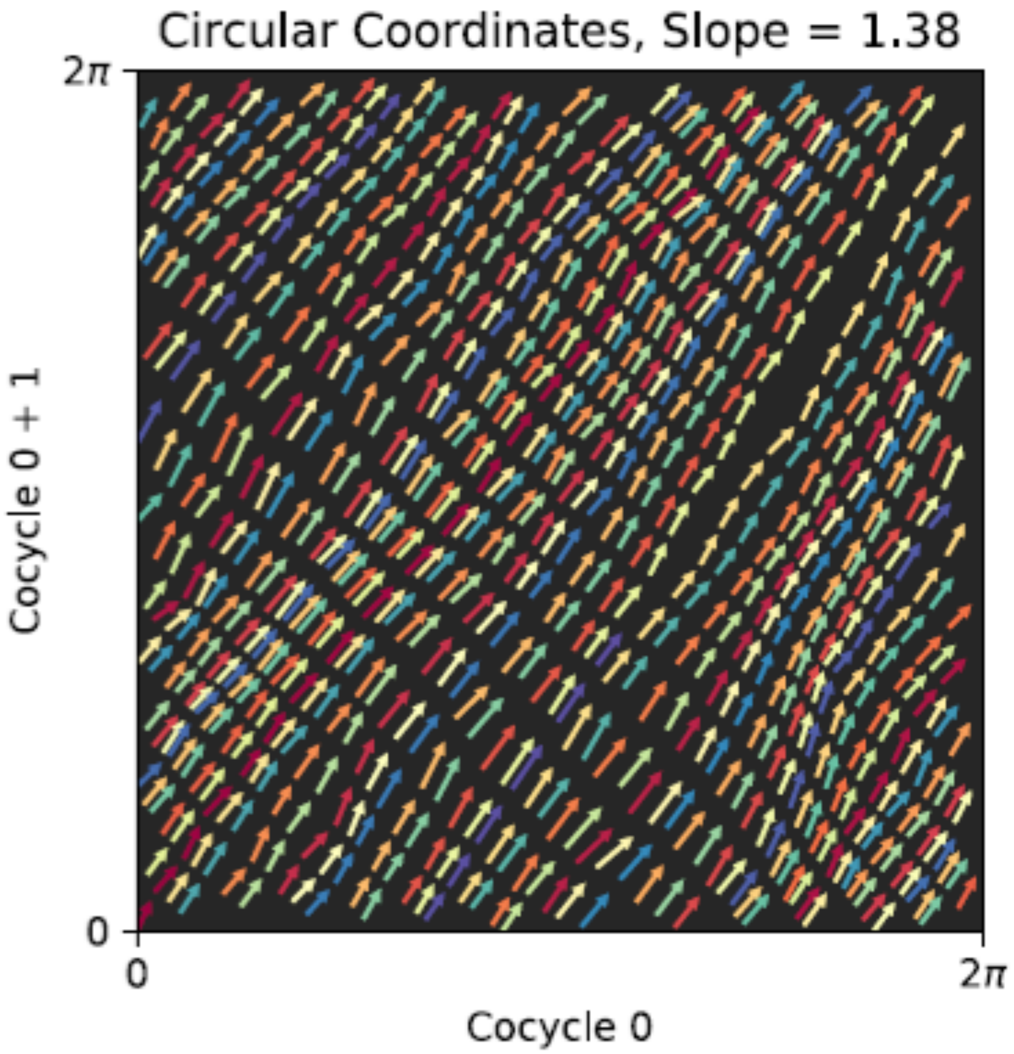
Circular Coordinates of Sliding Window Point Cloud

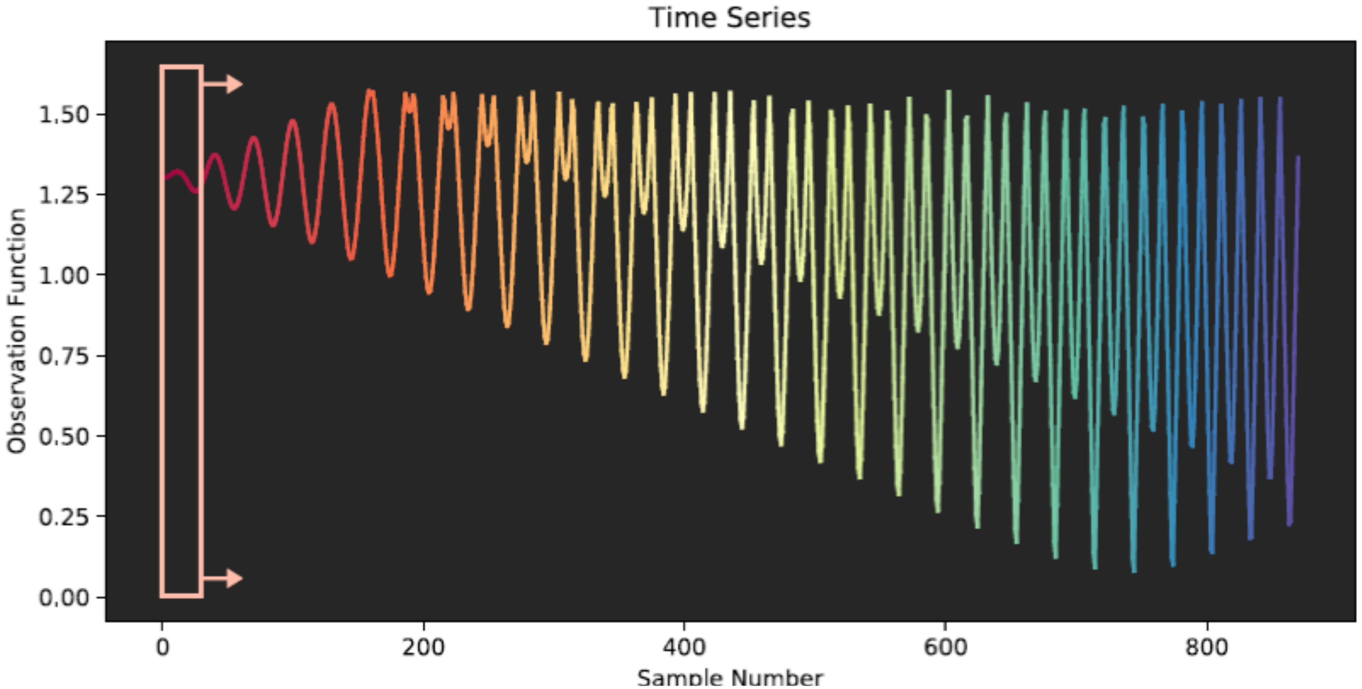


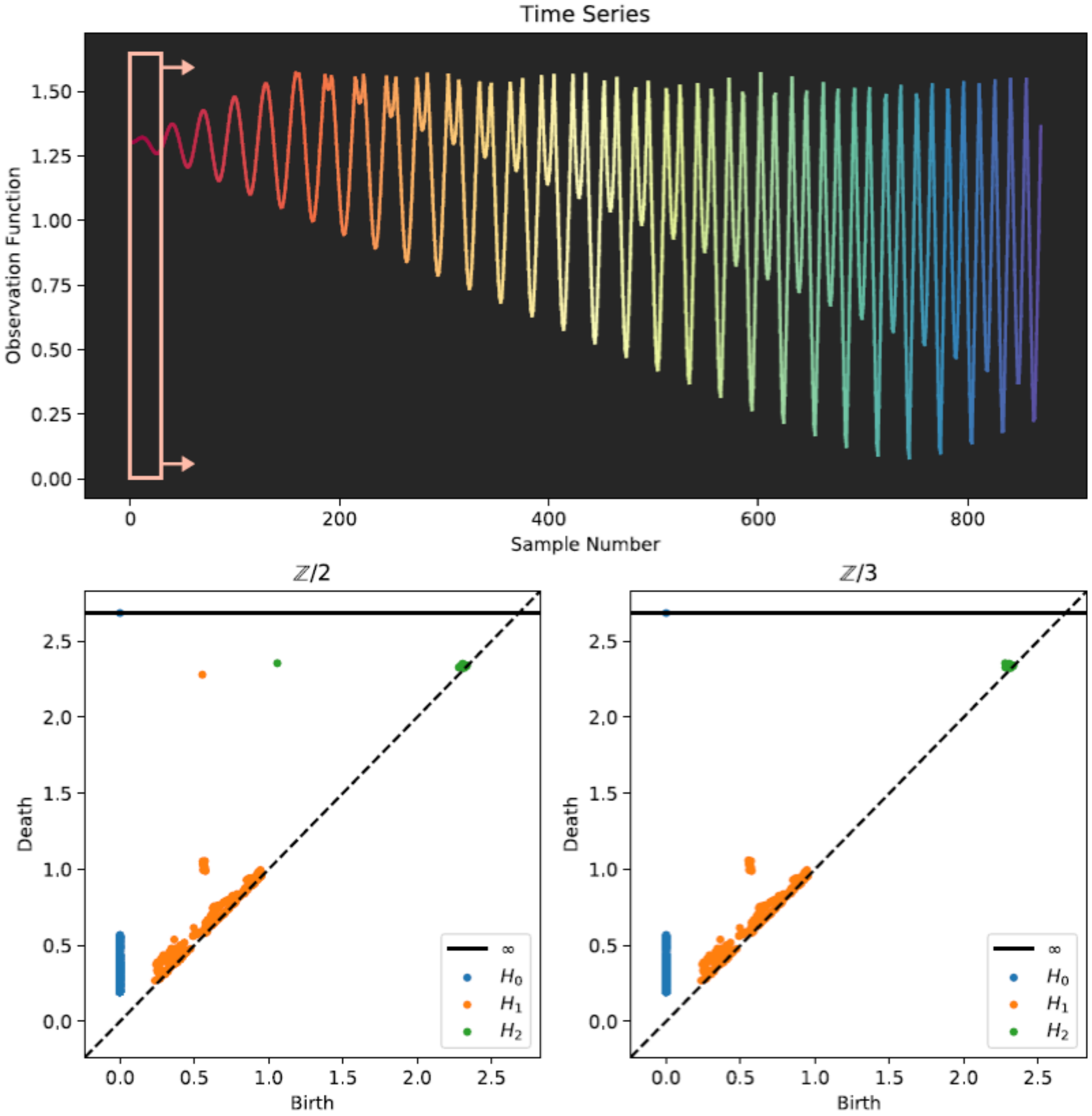
Original Dynamics

Recovered Dynamics



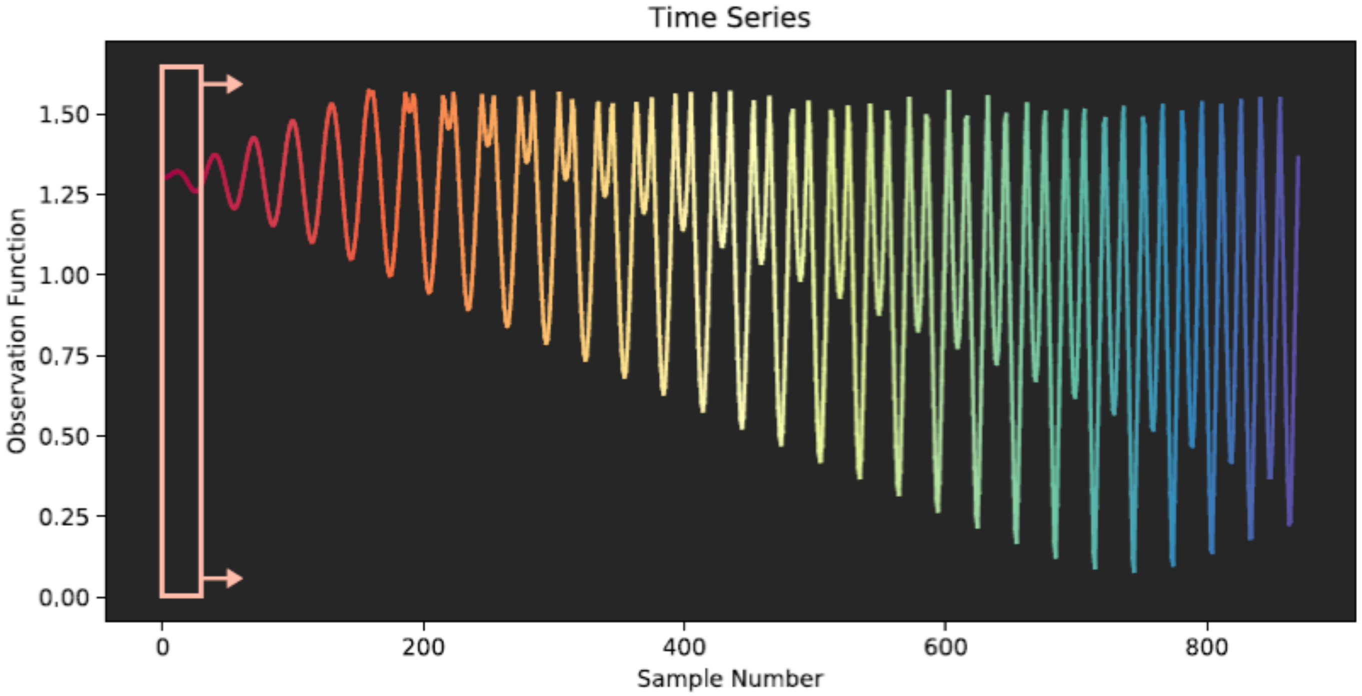


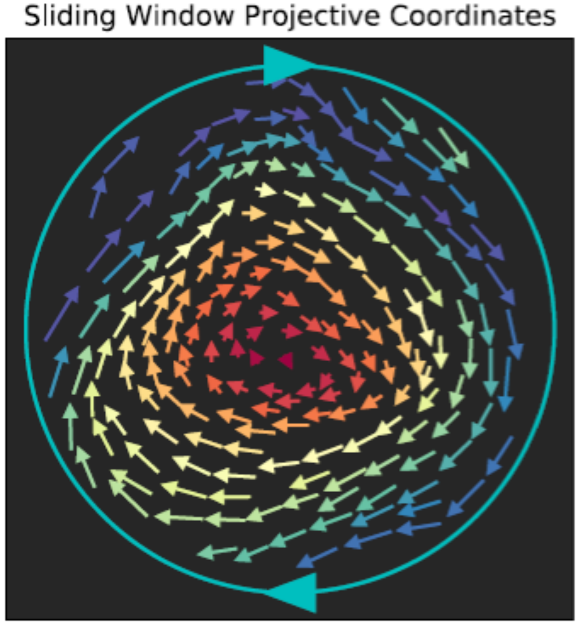

Time Series
Observation Function
Sample Number
1.50
1.25
1.00
0.75
0.50
0.25
0.00
0
200
400
600
800

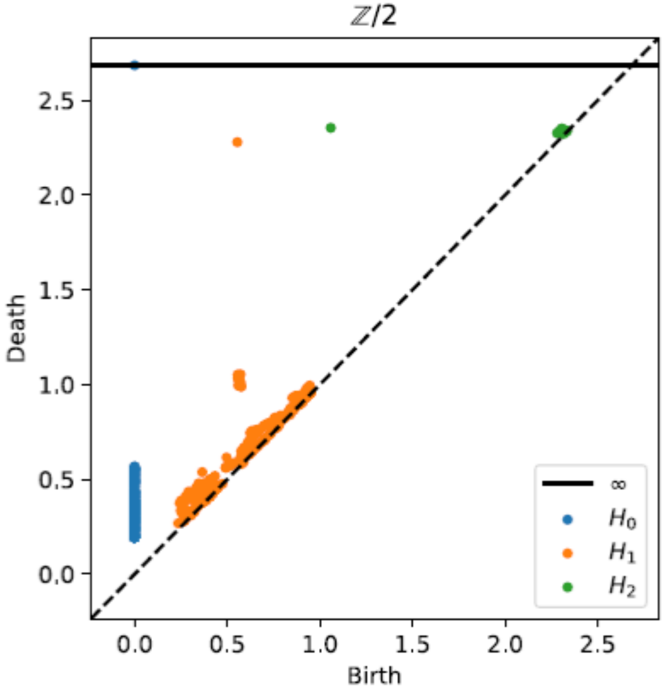


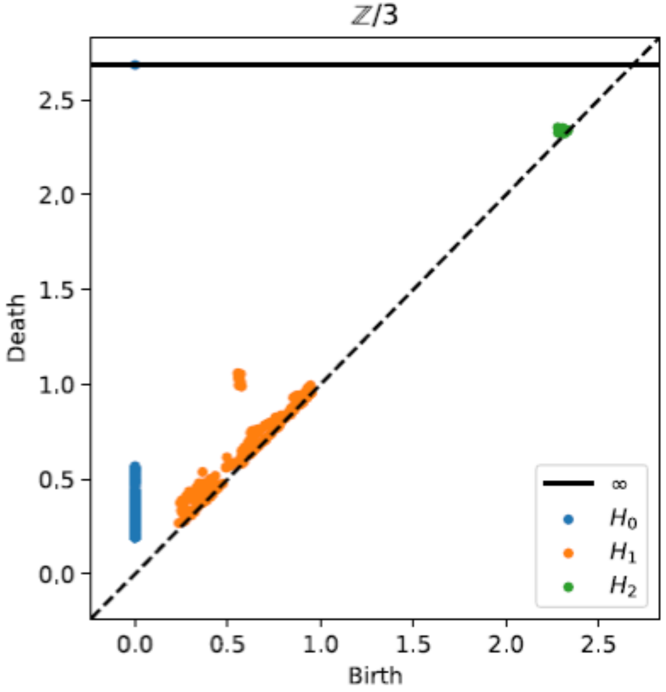
Time Series
Observation Function
Sample Number
ℤ/2
ℤ/3
Death
Birth
∞
H_0
H_1
H_2

Recovered Dynamics

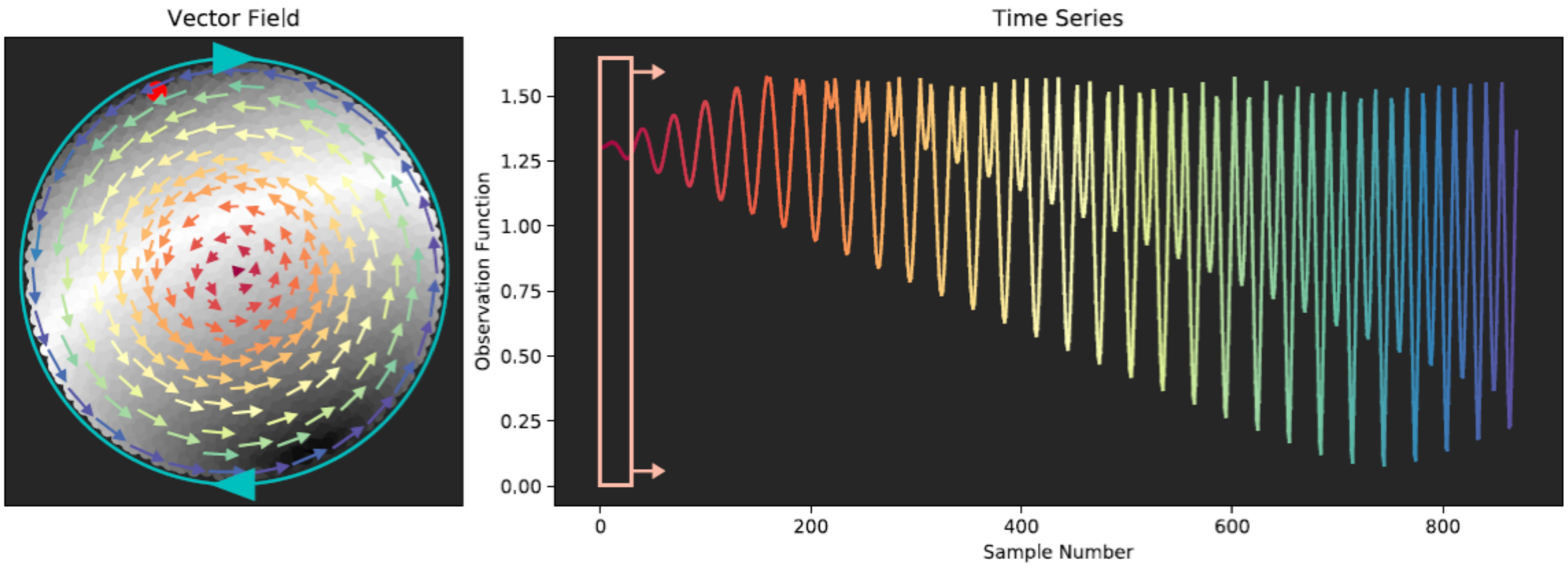




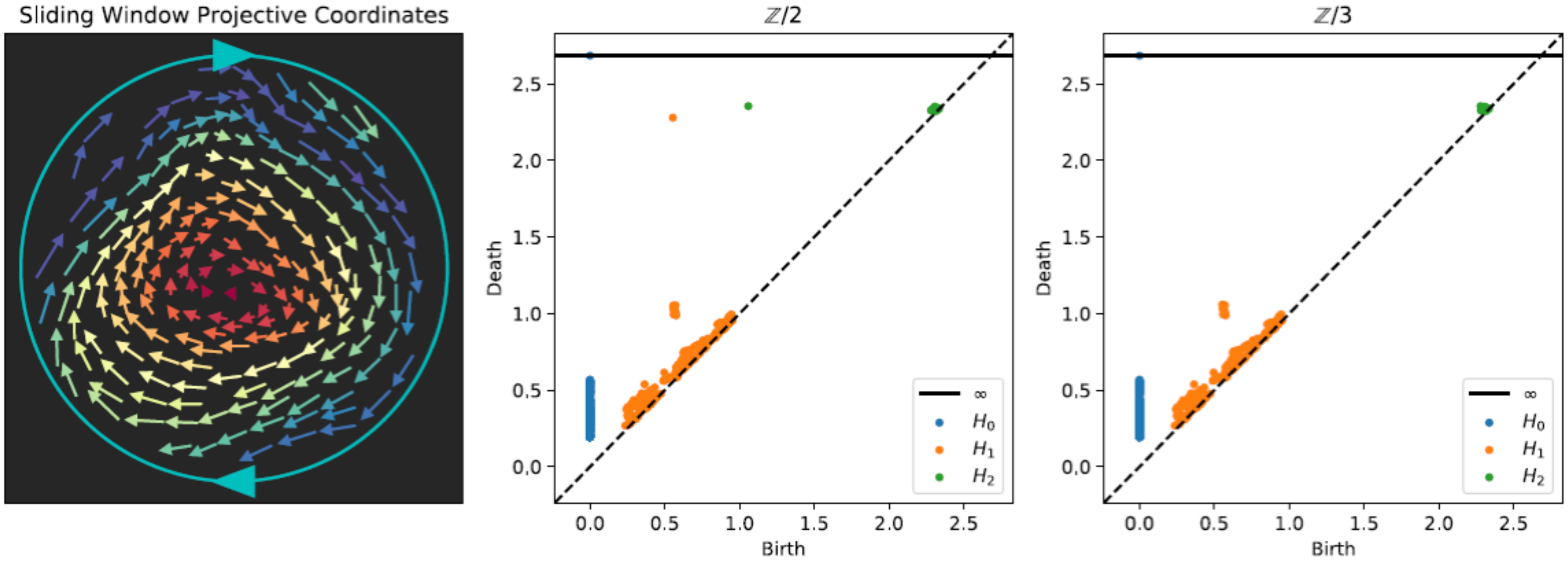


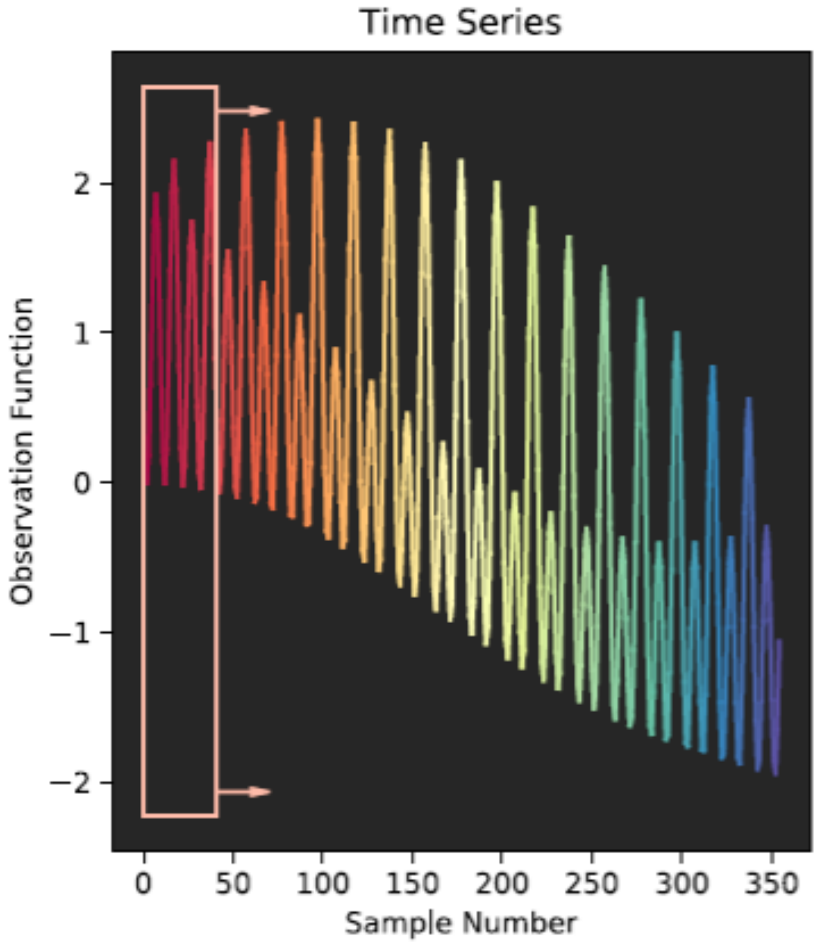


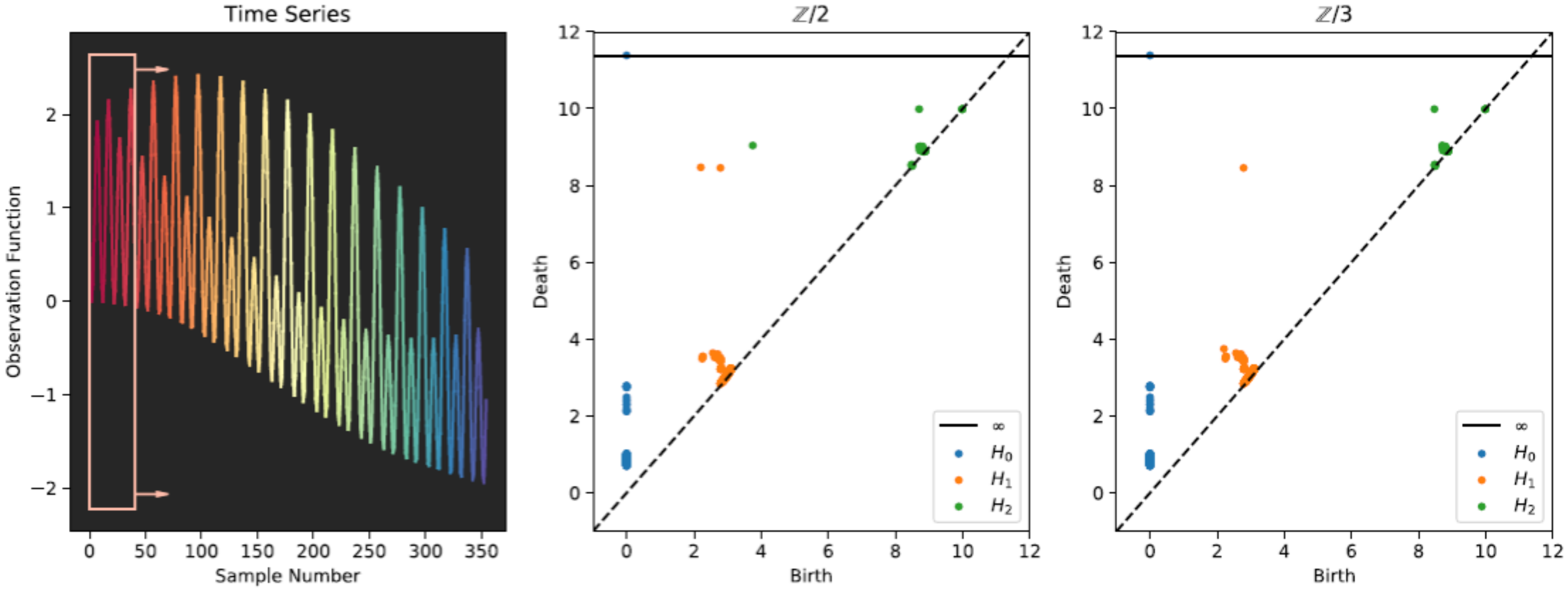
Original Dynamics



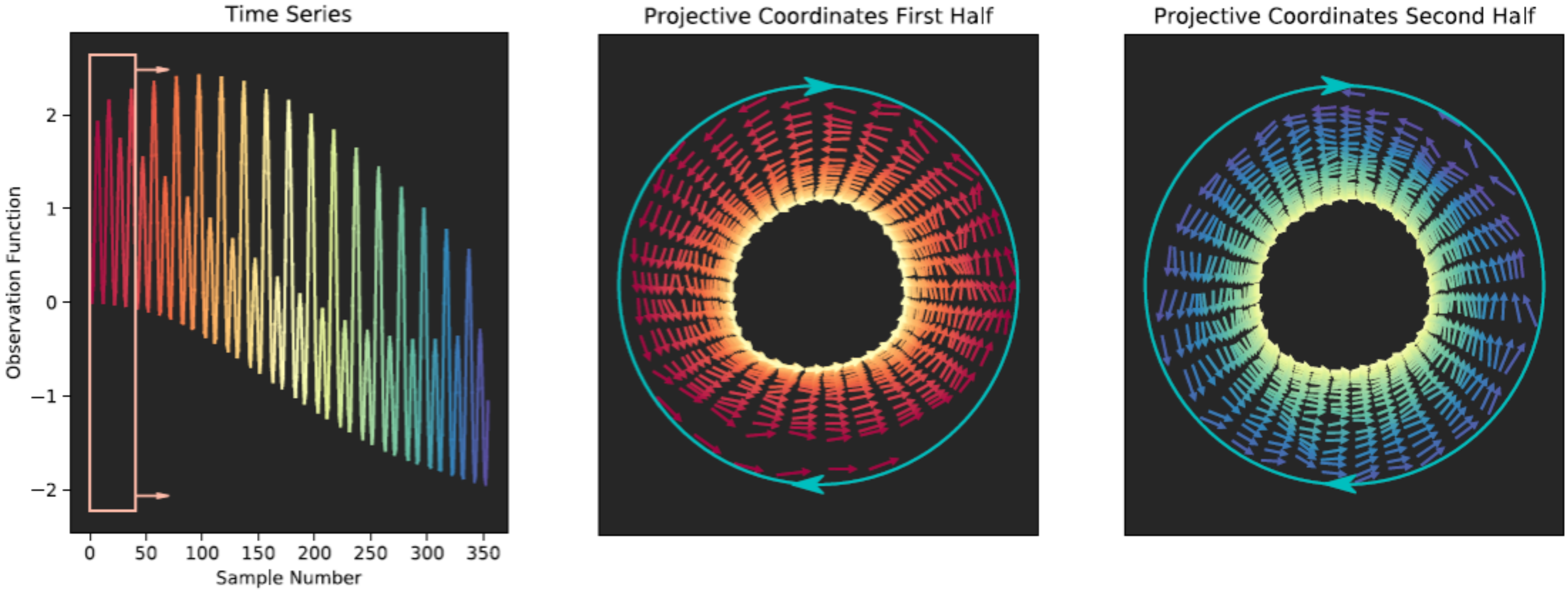
Recovered Dynamics




Time Series
Observation Function
Sample Number
2
1
0
−1
−2
0
50
100
150
200
250
300
350


Time Series
Observation Function
Sample Number
ℤ/2
ℤ/3
Death
Birth
∞
H_0
H_1
H_2

Projective Coordinates of Klein-Bottle Time Series



Code

- Today's demos: MATLAB wrapper of RIPSER + Cocycle representatives

- Python Library (w/ Chris Tralie):

  DREiMac: Dimension Reduction with Eilenberg-MacLane Coordinates

  https://github.com/ctralie/DREiMac

Thanks!

B. Xu, C. J. Tralie, A. Antia, M. Lin and J. A. Perea*, Twisty Takens: A Geometric Characterization of Good Observations on Dense Trajectories*, **Preprint**, 2018

J. A. Perea*, Towards Sparse, Stable and Transductive Circular Coordinates*, **Preprint**, 2018

J. A. Perea*, Multiscale Projective Coordinates via Persistent Cohomology of Sparse Filtrations*, **Discrete & Computational Geometry**, 2018

J. A. Perea, *Persistent Homology of Toroidal Sliding Window Embeddings,* **IEEE-ICASSP**, 2016

J. A. Perea and J. Harer, *Sliding Windows and Persistence: An Application of Topological Methods to Signal Analysis,* **Foundations of Computational Mathematics**, 2015