

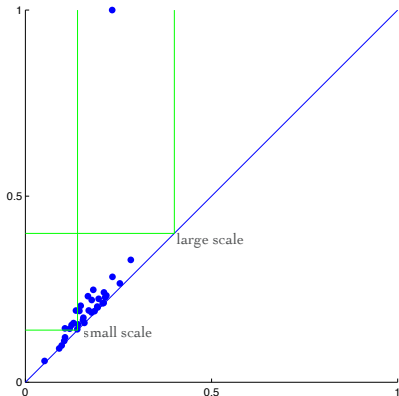
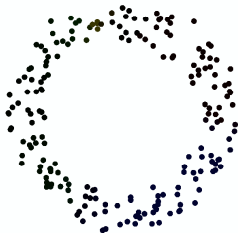
Operations on Reeb Graphs

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Story 1: Persistence diagrams

Persistent homology takes a filtered space $\mathbb{X} = \{X_t \mid t \in \mathbf{R}\}$ and returns a **barcode** of intervals $[p, q) \subset \mathbf{R}$ or a **persistence diagram** of points $(p, q) \in \mathbf{R}^2$.



Story 1: Persistence diagrams

Persistent homology takes a filtered space $\mathbb{X} = \{X_t \mid t \in \mathbf{R}\}$ and returns a **barcode** of intervals $[p, q) \subset \mathbf{R}$ or a **persistence diagram** of points $(p, q) \in \mathbf{R}^2$.

How is this defined?

Algorithmic approach (Edelsbrunner, Letscher, Zomorodian 2000).

- Discretize the t -variable.
- Present \mathbb{X} as a finite list of cells, attached in sequence.
- Some cells σ generate new homology cycles.
- Other cells τ destroy cycles created by an earlier σ .
- There is an interval $[t_\sigma, t_\tau)$ for each such pair (σ, τ) .
- There is an interval $[t_\sigma, +\infty)$ for each σ whose cycle is never destroyed.

Using commutative algebra (Zomorodian, Carlsson 2003).

- Discretize the t -variable to integers: $t = 0, 1, 2, \dots$
- Present \mathbb{X} as an increasing sequence:

$$\mathbb{X}: X_0 \subset X_1 \subset X_2 \subset \dots$$

- Apply a homology functor $H = H(-; \mathbf{k})$ to the sequence:

$$H(\mathbb{X}): H(X_0) \rightarrow H(X_1) \rightarrow H(X_2) \rightarrow \dots$$

- Observe that $H(\mathbb{X})$ is a graded module over the polynomial ring $\mathbf{k}[z]$, where z acts by shifting to the right.
- Decompose this graded module as a direct sum of cyclic submodules.
- Summands $z^s \mathbf{k}[z]/(z^{t-s})$ are recorded as intervals $[s, t)$.
- Summands $z^s \mathbf{k}[z]$ are recorded as intervals $[s, +\infty)$.

Using quiver theory (Carlsson, dS 2010).

- Discretize the t -variable to integers: $t = 0, 1, \dots, n - 1$.
- Present \mathbb{X} as a sequence of spaces with maps:

$$\mathbb{X}: X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_{n-1}$$

- Apply a homology functor $H = H(-; \mathbf{k})$ to the sequence:

$$H(\mathbb{X}): H(X_0) \rightarrow H(X_1) \rightarrow \cdots \rightarrow H(X_{n-1})$$

- Observe that $H(\mathbb{X})$ is a representation of the quiver $\bullet \rightarrow \bullet \rightarrow \dots \rightarrow \bullet$.
- Decompose $H(\mathbb{X})$ as a direct sum of indecomposable representations.
- According to Gabriel (1970), the indecomposables are precisely the **intervals**:

$$0 \rightarrow \cdots \rightarrow 0 \rightarrow \mathbf{k} \rightarrow \cdots \rightarrow \mathbf{k} \rightarrow 0 \rightarrow \cdots \rightarrow 0$$

The list of summands of $H(\mathbb{X})$ gives the persistence intervals.

When the arrows have mixed orientations \leftarrow, \rightarrow , we get **zigzag persistence**.

Story 2: Interleaving

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

The map $\{\text{persistence modules}\} \rightarrow \{\text{diagrams}\}$ is 1-Lipschitz.

Relators

The metrics on the two spaces are defined in terms of 'relators'.

- Two persistence modules may be related by an **interleaving**.
- Two diagrams may be related by a **matching**.

Every relator, of each type, has a size associated with it. The metrics are defined by finding the infimum of the size of relators between a given pair of objects. (Compare the geodesic distance in a Riemannian manifold.)

Stability theorem (Cohen-Steiner, Edelsbrunner, Harer 2007)

If two persistence modules admit an ϵ -interleaving, then their persistence diagrams admit an ϵ -matching.

Story 2: Interleaving

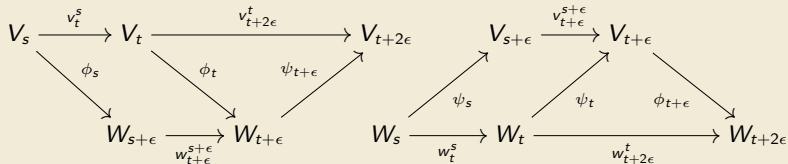
Definition

Let \mathbb{V}, \mathbb{W} be persistence modules. An ϵ -interleaving between \mathbb{V}, \mathbb{W} is a pair (Φ, Ψ) where $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ are collections of maps

$$\phi_t : V_t \rightarrow W_{t+\epsilon} \qquad \psi_t : W_t \rightarrow V_{t+\epsilon}$$

such that [various conditions].

The [various conditions] require the diagrams



to commute for all $s < t$.

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such that [various conditions].

The [various conditions] amount to the assertion that there is a unique way to get from any of the V_t, W_t to any other. All compositions of the $v_s^t, w_s^t, \phi_t, \psi_t$ with the same start and end point must agree.

Story 2: Interleaving

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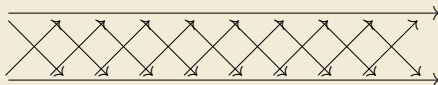
such that [various conditions].

Interleaver categories (Chazal, dS, Glisse, Oudot 2016)

An ϵ -interleaved pair of modules $(\mathbb{V}, \mathbb{W}, \Phi, \Psi)$ is 'the same thing' as a persistence module defined over the set $\mathbf{I} = \mathbf{R} \times \{0, \epsilon\}$ (two copies of the real line) with the partial order

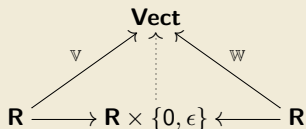
$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ s + \epsilon \leq t & \text{if } a \neq b \end{cases}$$

$\mathbf{R} \times \{0, \epsilon\}$:



Interleavings for classical persistence modules

Two classical persistence modules \mathbb{V}, \mathbb{W} are ϵ -interleaved iff the following functor extension problem has a solution:



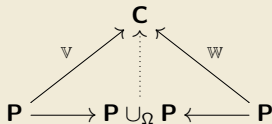
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Story 3: Interleaving Metrics

Interleavings for generalized persistence modules over a poset

Two persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \rightarrow \mathbf{C}$ are Ω -interleaved iff the following functor extension problem has a solution:



Here $\mathbf{P} \cup_{\Omega} \mathbf{P}$ has the partial order

$$(s, a) \leq (t, b) \Leftrightarrow \begin{cases} s \leq t & \text{if } a = b \\ \Omega s \leq t & \text{if } a \neq b \end{cases}$$

where $\Omega : \mathbf{P} \rightarrow \mathbf{P}$ is a **translation**.

Story 3: Interleaving Metrics

Translations (Bubenik, dS, Scott 2015)

$\mathbf{Trans}_{\mathbf{P}}$ is the poset of functions $\Omega : \mathbf{P} \rightarrow \mathbf{P}$ that are order-preserving and satisfy $x \leq \Omega x$ for all $x \in \mathbf{P}$.

Superlinear Families

A **superlinear family** is a 1-parameter family of translations of \mathbf{P}

$$(\Omega_{\epsilon} \mid \epsilon \in [0, \infty))$$

such that

$$\Omega_{\epsilon_1} \Omega_{\epsilon_2} \leq \Omega_{\epsilon_1 + \epsilon_2}$$

for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Sublinear Projections

A **sublinear projection** is a map $\pi : \mathbf{Trans}_{\mathbf{P}} \rightarrow [0, \infty]$ such that

$$\pi(\Omega_1 \Omega_2) \leq \pi(\Omega_1) + \pi(\Omega_2)$$

for all $\Omega_1, \Omega_2 \in \mathbf{Trans}_{\mathbf{P}}$.

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Examples of superlinear families

- $\mathbf{P} = \mathbf{R}$,
 $\Omega_\epsilon(t) = t + \epsilon$.
- $\mathbf{P} = \{\text{compact intervals in the real line}\}$,
 $\Omega_\epsilon([a, b]) = [a - \epsilon, b + \epsilon]$.
- $\mathbf{P} = \{\text{closed subsets of a metric space } X\}$,
 $\Omega_\epsilon(V) = V^\epsilon = \{x \in X \text{ such that } d(x, V) \leq \epsilon\}$.

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for all $\epsilon_1, \epsilon_2 \in [0, \infty)$.

Interleaving distance (Bubenik, dS, Scott 2015)

Given a superlinear family (Ω_ϵ) of translations of \mathbf{P} , we define the interleaving distance

$$d_i(\mathbb{V}, \mathbb{W}) = \inf(\epsilon \mid \mathbb{V}, \mathbb{W} \text{ are } \Omega_\epsilon\text{-interleaved})$$

between generalized persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{P} \rightarrow \mathbf{C}$.

Story 3: Interleaving Metrics

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Story 3: Interleaving Metrics

Functoriality

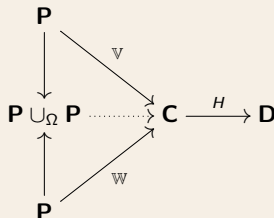
Suppose $\mathbb{V}, \mathbb{W} : \mathbf{P} \rightarrow \mathbf{C}$ and $H : \mathbf{C} \rightarrow \mathbf{D}$ are functors. Then

$$d_i(H\mathbb{V}, H\mathbb{W}) \leq d_i(\mathbb{V}, \mathbb{W})$$

for any superlinear family or sublinear projection.

Proof.

An Ω -interleaving of \mathbb{V}, \mathbb{W} gives an Ω -interleaving of $H\mathbb{V}, H\mathbb{W}$:



Story 3: Interleaving Metrics

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Example: sublevelset persistence

Let X be a topological space and $f, g : X \rightarrow \mathbf{R}$ be functions with $\|f - g\|_\infty \leq \epsilon$.

- The persistence modules $\mathbb{V}, \mathbb{W} : \mathbf{R} \rightarrow \mathbf{Top}$ defined

$$\mathbb{V}(t) = f^{-1}(-\infty, t], \quad \mathbb{W}(t) = g^{-1}(-\infty, t],$$

are ϵ -interleaved.

(There are natural inclusions $\mathbb{V}(t) \subseteq \mathbb{W}(t + \epsilon)$ and $\mathbb{W}(t) \subseteq \mathbb{V}(t + \epsilon)$.)

- For any homology functor $H : \mathbf{Top} \rightarrow \mathbf{Vect}$, the persistence modules $H\mathbb{V}, H\mathbb{W} : \mathbf{R} \rightarrow \mathbf{Vect}$ are ϵ -interleaved.

Story 4: Set-Valued Persistence Modules

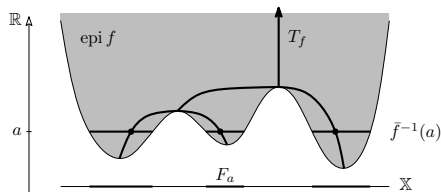
Merge trees (Cagliari, Ferri, Pozzi 2001, & Morozov, Beketayev, Weber 2013)

- A functor $\mathbf{T} : \mathbf{R} \rightarrow \mathbf{Set}$ can be thought of as a **merge tree**.
- Let X be a topological space and $f : X \rightarrow \mathbf{R}$ a function. Then

$$\mathbf{T}(t) = \pi_0 f^{-1}(-\infty, t]$$

$$\mathbf{T}[s \leq t] = \pi_0 [f^{-1}(-\infty, s] \subseteq f^{-1}(-\infty, t]]$$

defines the **sublevelset merge tree** of (X, f) .



- If $f, g : X \rightarrow \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $d_i(\mathbf{T}, \mathbf{U}) \leq \epsilon$.

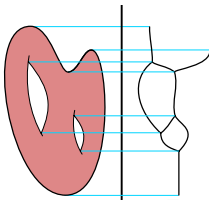
Story 4: Set-Valued Persistence Modules

Reeb graphs (dS, Munch, Patel 2016)

- A functor $\mathbf{F} : \mathbf{Int} \rightarrow \mathbf{Set}$ can be thought of as a **graph over the real line**. (Technically we require \mathbf{F} to satisfy a **cosheaf condition**.)
- Let X be a topological space and $f : X \rightarrow \mathbf{R}$ a function. Then

$$\mathbf{F}_f(I) = \pi_0 f^{-1}(I)$$
$$\mathbf{F}_f[I \subseteq J] = \pi_0 [f^{-1}(I) \subseteq f^{-1}(J)]$$

defines the **Reeb graph** of (X, f) .

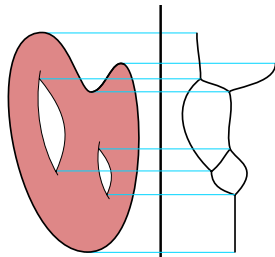


- If $f, g : X \rightarrow \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $d_i(\mathbf{F}, \mathbf{G}) \leq \epsilon$.

Story 5: Reeb Graphs & Reeb Cosheaves

Reeb graphs

- An **R-space** (X, f) is a topological space X with a map $f : X \rightarrow \mathbf{R}$.
- An **R-space** is a **Reeb graph** if each $f^{-1}(t)$ is finite.



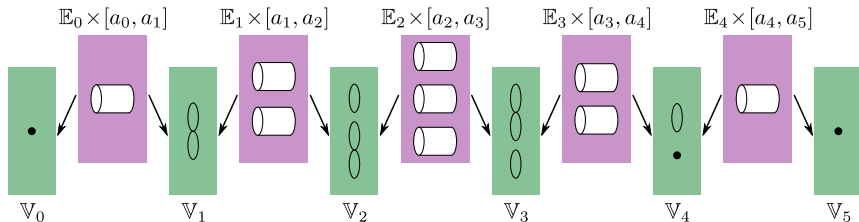
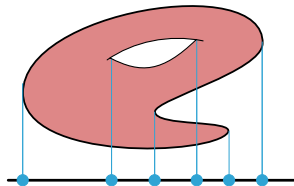
Reeb functor

- The **Reeb functor** converts a (constructible) **R-space** into a Reeb graph:

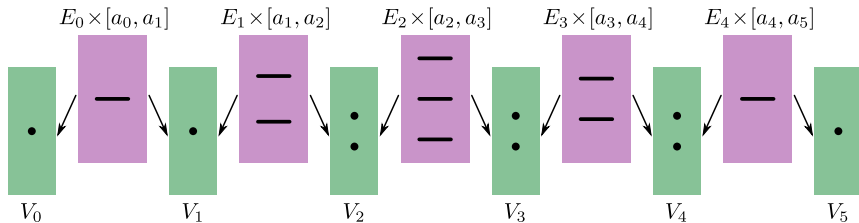
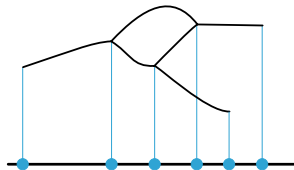
$$(X, f) \longmapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff x, y are in the same component of the same levelset of f .

Story 5: Reeb Graphs & Reeb Cosheaves



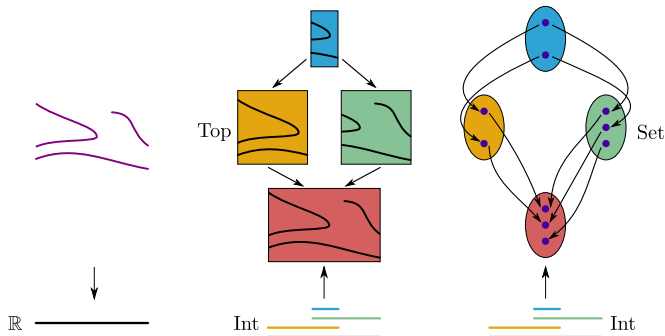
Story 5: Reeb Graphs & Reeb Cosheaves



Story 5: Reeb Graphs & Reeb Cosheaves

Reeb cosheaves (dS, Munch, Patel 2016)

- Let **Int** denote the poset of open intervals, with respect to inclusion.
- A Reeb graph gives rise to a functor $\mathbf{F} : \mathbf{Int} \rightarrow \mathbf{Set}$ that is **constructible** and satisfies the **cosheaf condition** for unions of intervals.

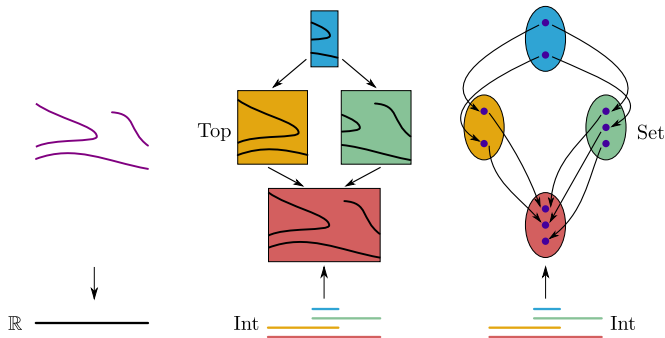


$$\mathbf{F}(I = \bigcup I_\alpha) = \operatorname{colim} \left[\coprod_{\alpha, \beta} \mathbf{F}(I_\alpha \cap I_\beta) \rightrightarrows \coprod_{\alpha} \mathbf{F}(I_\alpha) \right]$$

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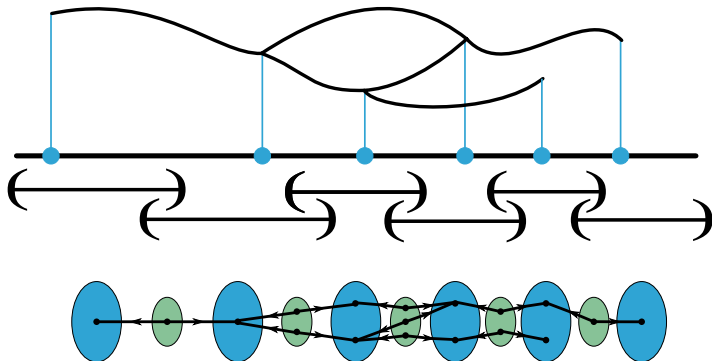


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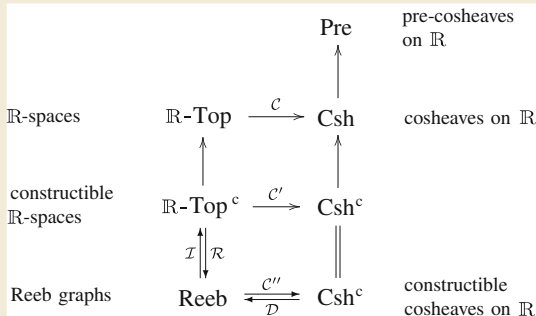
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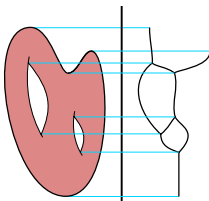


Story 5: Reeb Graphs & Reeb Cosheaves

Reeb cosheaves (dS, Munch, Patel 2016)

- Let **Int** denote the poset of open intervals, with respect to inclusion.
- A Reeb graph **corresponds** to a functor $\mathbf{F} : \mathbf{Int} \rightarrow \mathbf{Set}$ that is **constructible** and satisfies the **cosheaf condition** for unions of intervals.





Reeb functor (two versions)

- The **Reeb functor** converts a (constructible) \mathbf{R} -space into a Reeb graph:

$$(X, f) \mapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff x, y are in the same component of the same levelset of f .

or

- The **Reeb functor** converts a constructible \mathbf{R} -space into a Reeb cosheaf:

$$\mathbf{F}(I) = \pi_0 f^{-1}(I)$$

$$\mathbf{G}[I \subseteq J] = \pi_0 [f^{-1}(I) \subseteq f^{-1}(J)]$$

Story 5: Reeb Graphs & Reeb Cosheaves

Translation operators on \mathbf{Int}

We define a 1-parameter semigroup (Ω_ϵ) of functors $\mathbf{Int} \rightarrow \mathbf{Int}$ by setting

$$\Omega_\epsilon(I) = I^\epsilon = \text{“}\epsilon\text{-neighbourhood of } I\text{”}$$

Reeb interleaving distance (dS, Munch, Patel 2016)

An ϵ -**interleaving** between \mathbf{F}, \mathbf{G} is given by two families of maps

$$\phi_I : \mathbf{F}(I) \rightarrow \mathbf{G}(I^\epsilon), \quad \psi_I : \mathbf{G}(I) \rightarrow \mathbf{F}(I^\epsilon)$$

which are natural with respect to inclusions $I \subseteq J$, and such that for all I

$$\psi_{I^\epsilon} \circ \phi_I = \mathbf{F}[I \subseteq I^{2\epsilon}], \quad \phi_{I^\epsilon} \circ \psi_I = \mathbf{G}[I \subseteq I^{2\epsilon}].$$

Stability Theorem

If $f, g : X \rightarrow \mathbf{R}$ with $\|f - g\|_\infty \leq \epsilon$ then $d_i(\mathbf{F}, \mathbf{G}) \leq \epsilon$.

Universal Reeb Metric (Bauer, Landi, Mémoli 2018)

The **universal metric** $d_u(\mathbf{F}, \mathbf{G})$ is the largest that satisfies the stability theorem.

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Cosheaf Smoothing Theorem

If $\mathbf{F} : \mathbf{Int} \rightarrow \mathbf{Set}$ is a (constructible) cosheaf, then so is $\mathbf{F}\Omega_\epsilon : \mathbf{Int} \rightarrow \mathbf{Set}$.

Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.

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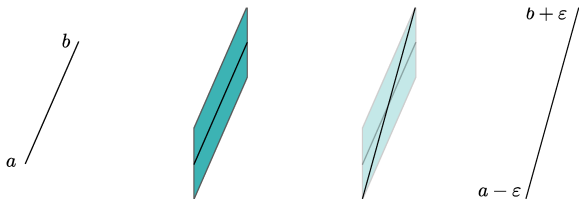
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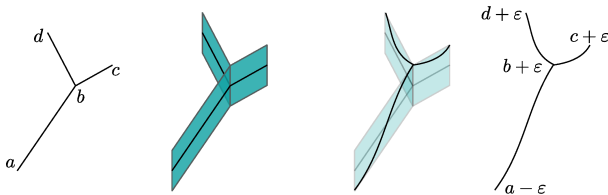
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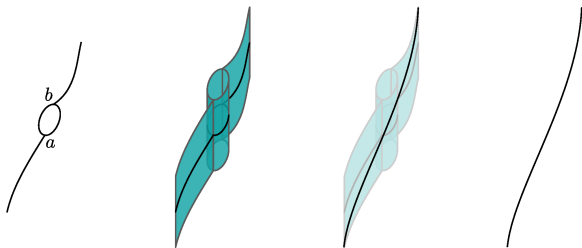
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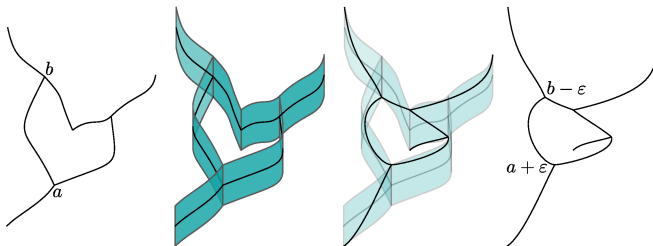
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Cosheaf Smoothing Theorem

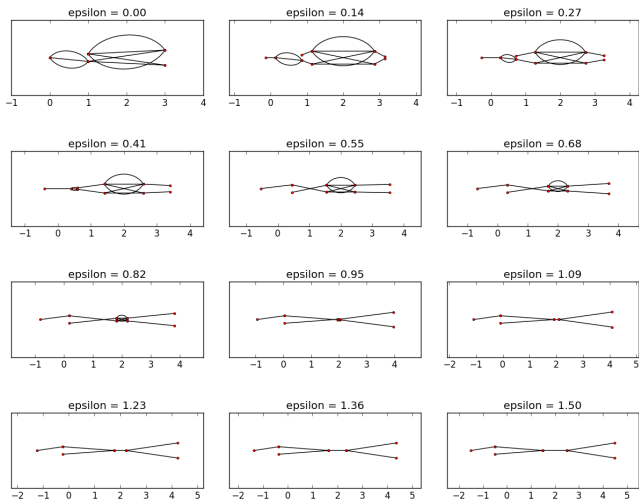
If $\mathbf{F} : \mathbf{Int} \rightarrow \mathbf{Set}$ is a (constructible) cosheaf, then so is $\mathbf{F}\Omega_\epsilon : \mathbf{Int} \rightarrow \mathbf{Set}$.

Corollary: Reeb Smoothing

There is a 1-parameter semigroup of ‘smoothing’ operations on Reeb graphs.



Story 6: Reeb smoothing algorithm by Dmitriy Smirnov & Song Yu



Discretized Reeb Graphs

- A discrete Reeb graph is a diagram

$$E \begin{array}{c} \xrightarrow{\ell} \\ \xrightarrow{r} \end{array} V \xrightarrow{\phi} \mathbf{R}$$

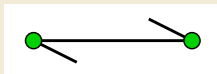
where E, V are finite sets and $\phi\ell(e) < \phi r(e)$ for each $e \in E$.

- Each $v \in V$ has a **left-** and **right-degree**:

$$\deg_l(v) = \#r^{-1}(v), \quad \deg_r(v) = \#\ell^{-1}(v), \quad \deg(v) = (\deg_l(v), \deg_r(v)).$$

- The discrete Reeb graph is **reduced** if $\deg(v) \neq (1, 1)$ for all v .

The **critical radius** is half the length of the smallest 'hook':



$$\epsilon_{crit} = \frac{1}{2} \min \{ \phi r(e) - \phi \ell(v) \mid e \in E, \deg_r(\ell(e)) > 1, \deg_l(r(e)) > 1 \}$$

Algorithm: smooth by ϵ

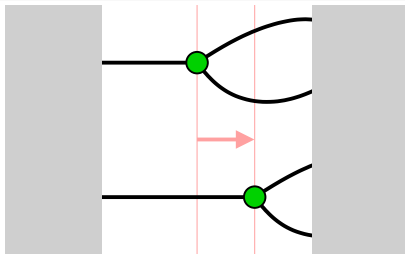
- If $\deg(v) = (1, ?)$ then v moves by $+\epsilon$.
- If $\deg(v) = (?, 1)$ then v moves by $-\epsilon$.
- If $\deg(v) = (?, ?)$ then split v into two and move by $\pm\epsilon$.

Valid up to the critical radius. Recompute at critical radius and recurse.

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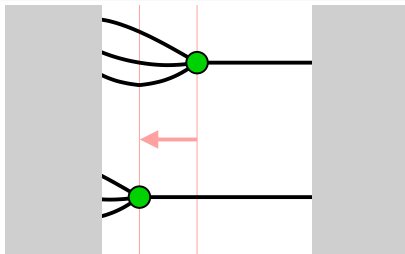


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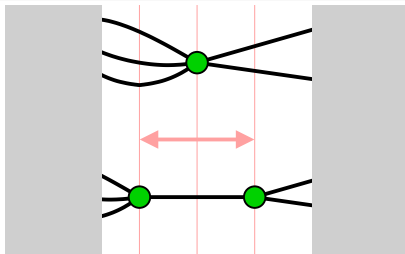


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Story 6: Reeb smoothing algorithm by Dmitriy Smirnov & Song Yu

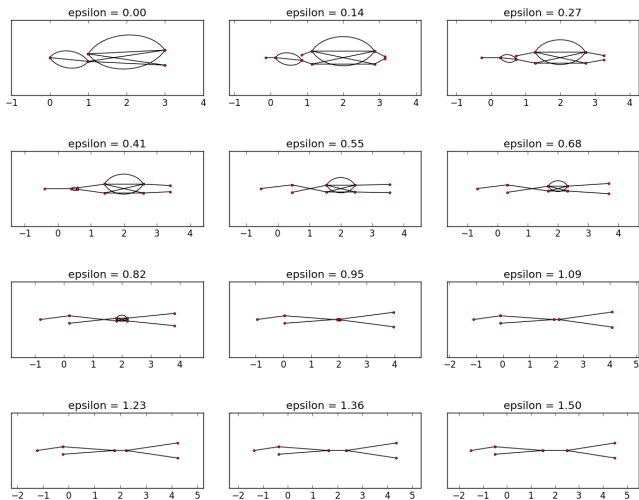
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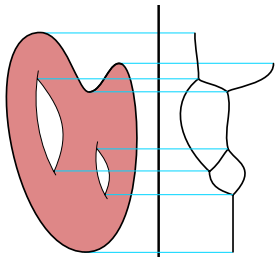
Story 6: Reeb smoothing algorithm by Dmitriy Smirnov & Song Yu



Story 7: Reeb space operations

Reeb graphs

- An **R-space** (X, f) is a topological space X with a map $f : X \rightarrow \mathbf{R}$.
- An **R-space** is a **Reeb graph** if X is a graph and each $f^{-1}(t)$ is finite.



Reeb functor

- The **Reeb functor** converts a (constructible) **R-space** into a Reeb graph:

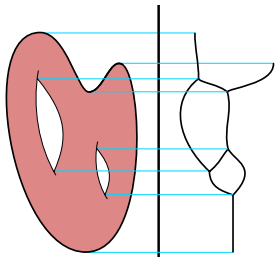
$$(X, f) \longmapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff x, y are in the same component of the same levelset of f .

Story 7: Reeb space operations

Reeb spaces

- A **B-space** (X, f) is a topological space X with a map $f : X \rightarrow \mathbf{B}$.
- A **B-space** is a **Reeb B-space** if each $f^{-1}(t)$ is finite.



Reeb functor

- The **Reeb functor** converts a (constructible) **B-space** into a Reeb **B-space**:

$$(X, f) \mapsto ((X/\sim), \bar{f})$$

where $x \sim y$ iff x, y are in the same component of the same **fiber** of f .

Example: Universal Cover

Let \mathbf{B} be a (locally well-behaved) topological space. Then

$$\text{Path}(\mathbf{B}, b_0) = \{\text{paths } \gamma : [0, 1] \rightarrow \mathbf{B} \text{ with } \gamma(0) = b_0\}$$

is a \mathbf{B} -space with respect to the evaluation map

$$e : \text{Path}(\mathbf{B}, b_0) \longrightarrow \mathbf{B}; \gamma \longmapsto \gamma(1).$$

Then

$$\text{Univ}(\mathbf{B}) = \text{Reeb} [\text{Path}(\mathbf{B}, b_0), e]$$

is the universal cover of \mathbf{B} .

Example: Universal Reeb Metric (Bauer, Landi, Mémoli 2018)

Let $X = (X, f)$ and $Y = (Y, g)$ be Reeb graphs.

- A **relator** for X, Y is an $(\mathbf{R} \times \mathbf{R})$ -space

$$W \xrightarrow{F} \mathbf{R} \times \mathbf{R}$$

such that

$$\text{Reeb} [W, p_1 \circ F] \cong (X, f),$$

$$\text{Reeb} [W, p_2 \circ F] \cong (Y, g).$$

- The **deviation** of a relator

$$\text{dev}(W) = \sup_{w \in W} |p_1(F(w)) - p_2(F(w))|$$

measures how far $F(W)$ deviates from the diagonal.

- The **universal distance** between X, Y is defined

$$d_u(X, Y) = \inf (\text{dev}(W) \mid W \text{ is a relator for } X, Y)$$

Story 7: Reeb space operations

Let \mathbf{B} be a topological semigroup with operation \odot .

Reeb \mathbf{B} -space convolutions

The π_0 -**convolution** of Reeb spaces

$$X \xrightarrow{f} \mathbf{B}, \quad Y \xrightarrow{g} \mathbf{B}$$

is defined to be

$$(X, f) * (Y, g) = \text{Reeb} [X \times Y, f \odot g]$$

Reeb graph convolutions

$$(X, f) * (Y, g) = \text{Reeb} [X \times Y, f + g]$$

Reeb graph convolutions

$$(X, f) * (Y, g) = \text{Reeb} [X \times Y, f + g]$$

Examples

- The σ -smoothing of a Reeb graph $X = (X, f)$ is given by the formula

$$X^\sigma = X * [-\sigma, \sigma].$$

- The intervals $[-\sigma, \sigma]$, for $\sigma \geq 0$, form a semigroup under $*$.
- More generally, the convolution of intervals is their Minkowski sum:

$$[m_1 - \sigma_1, m_1 + \sigma_1] * [m_2 - \sigma_2, m_2 + \sigma_2] = [m - \sigma, m + \sigma]$$

where $m = m_1 + m_2$ and $\sigma = \sigma_1 + \sigma_2$.

- The **merge-tree** and **split-tree** of X are given by the formulas

$$\text{Merge}(X) = X * [0, +\infty), \quad \text{Split}(X) = X * (-\infty, 0].$$

- Thus $X * [-R, R]$, when $R \gg 0$, combines the merge and split trees of X .

Metric properties of Reeb graph smoothing

Let X, Y be Reeb graphs. Then

$$d_i(X^\sigma, Y^\sigma) \leq d_i(X, Y), \quad d_u(X^\sigma, Y^\sigma) \leq d_u(X, Y)$$

and

$$d_i(X, X^\sigma) \leq d_u(X, X^\sigma) \leq \sigma$$

for all $\sigma \geq 0$.

Analogy: Gaussian kernel smoothing

Is there a theory of π_0 signal processing?

Collaborators

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